Auxiliary materials for lecture 3

- Helicity amplitudes for the process $e^+e^- \to \mu^+\mu^-$
- Cross section $e^+e^- \to$ hadrons. $R(s)$ ratio.

The calculation of decay widths and scattering cross sections using helicity amplitudes is widely used along with taking traces. The method of helicity amplitudes makes it possible to significantly simplify analytical calculations.

Reminder: solutions of the Dirac equation for a particle and an antiparticle (standard matrix representation γ).

Dirac spinors for particle $\psi_{1,2} = u_{1,2}e^{i(\mathbf{px}-Et)}$

$$u_1(p) = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \ u_2(p) = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

Dirac spinors for antiparticle $\psi_{1,2} = v_{1,2}e^{-i(\mathbf{px}-Et)}$

$$v_1(p) = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 0 \\ 1 \end{pmatrix}, \ v_2(p) = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Calculating $\psi^+\psi = |N|^2\frac{2E}{E+m}$, we get $N = \sqrt{E+m}$, wavefunction normalisation to $2E$ particles in the unit volume. Four solutions for particle and antiparticle with zero momentum

$$u_{1,2}(p) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_{1,2}(p) = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

which are an eigenvectors of the spin operator

$$S_z = \frac{1}{2} \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and are spin up (+) and spin down (-) states. For nonzero momentum the states $u_{1,2}$ and $v_{1,2}$ are not S_z eigenvectors. Such eigenvectors are the states with momentum collinear to Z-axis, $p = |\mathbf{p}| = p_z$

$$u_{1,2}(p) = N \begin{pmatrix} 1 \\ 0 \\ \frac{\pm p}{E+m} \\ 0 \end{pmatrix}, N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp p}{E+m} \end{pmatrix},$$

$$v_{1,2}(p) = N \begin{pmatrix} 0 \\ \frac{\mp p}{E+m} \\ 0 \\ 1 \end{pmatrix}, N \begin{pmatrix} \frac{\pm p}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

For example

$$S_z u_1(E,0,0,\pm p) = +\tfrac{1}{2} u_1(E,0,0,\pm p)$$
$$S_z u_2(E,0,0,\pm p) = -\tfrac{1}{2} u_2(E,0,0,\pm p)$$

and so on. For particle/antiparticle with momentum $\mathbf{p} = (0,0,\pm p)$ the states u_1,v_1 have spin up while the states u_2,v_2 have spin down.

In the general case, for the analysis of cross sections in terms of spin states of particles, helicity (projection of the spin vector on the direction of momentum) and the corresponding helicity operator are introduced

$$H = \tfrac{\mathbf{s}\cdot\mathbf{p}}{|\mathbf{p}|}, \qquad \hat{H} = \tfrac{1}{2p}\begin{pmatrix} \sigma\cdot\mathbf{p} & 0 \\ 0 & \sigma\cdot\mathbf{p} \end{pmatrix}$$

The eigenvalues of the helicity operator are +1/2 (right helicity state) or -1/2 (left helicity state). Writing the vector $\mathbf{p}$ in spherical coordinates

$$\mathbf{p} = (p\sin\theta\cos\varphi, p\sin\theta\sin\varphi, p\cos\theta)$$

we find the eigenvalues and eigenvectors of the helicity operator

for the left/right spinor of the particle $(H = -1/2)$

$$u_R(p) = N \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \\ \frac{p}{E+m}\cos\frac{\theta}{2} \\ \frac{p}{E+m}e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}, u_L(p) = N \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2} \\ \frac{p}{E+m}\sin\frac{\theta}{2} \\ \frac{-p}{E+m}e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix}$$

for the left/right spinor of the antiparticle ($H = 1/2$)

$$v_R(p) = N \begin{pmatrix} \frac{p}{E+m}\sin\frac{\theta}{2} \\ \frac{-p}{E+m}e^{i\varphi}\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix}, v_L(p) = N \begin{pmatrix} \frac{p}{E+m}\cos\frac{\theta}{2} \\ \frac{p}{E+m}e^{i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

where $N = \sqrt{E+m}$. Mass is usually neglected in comparison with the energy.

To calculate the cross section of the process $e^+e^- \to \mu^+\mu^-$, it is necessary to calculate the fermion currents of the initial state and the final state by summing up all possible helicity states. For example, if both initial fermions are right (right helicity), we have

$\sum |M_{RR}|^2 = |M_{RR\to RR}|^2 + |M_{RR\to LR}|^2 + |M_{RR\to RL}|^2 + |M_{RR\to LL}|^2$

If there is no initial state polarization, the full matrix element

$$|M|^2 = |M_{RR}|^2 + |M_{LR}|^2 + |M_{RL}|^2 + |M_{LL}|^2$$

there are 16 helicity amplitudes in total. Since the helicity states are orthogonal, there is no interference of helicity amplitudes. We calculate the helicity amplitudes using the following parametrization of the vectors of (anti)particles in the initial and final states

$$p_1 = (E, 0, 0, E)$$
$$p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

and neglecting the fermion mass

$$u_R(p) = \sqrt{E}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}, u_L(p) = \sqrt{E}\begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$v_R(p) = \sqrt{E}\begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix}, v_L(p) = \sqrt{E}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

Then for components, for example, RL muon current, we get ($s = \sin\frac{\theta}{2}, c = \cos\frac{\theta}{2}$)

$j^0 = \bar{u}_R(p_3)\gamma^0 v_L(p_4) = E(cs - sc + cs - sc) = 0$
$j^1 = \bar{u}_R(p_3)\gamma^1 v_L(p_4) = E(-c^2 + s^2 - c^2 + s^2)$
$= 2E(s^2 - c^2) = -2E\cos\theta$
$j^2 = \bar{u}_R(p_3)\gamma^2 v_L(p_4) = -iE(-c^2 - s^2 - c^2 - s^2)$
$= 2E(s^2 - c^2) = 2iE$
$j^3 = \bar{u}_R(p_3)\gamma^3 v_L(p_4) = E(cs + sc + cs + sc)$
$= 4Esc = 2E\sin\theta$

so $j_{RL} = 2E(0, -\cos\theta, i, \sin\theta)$. Analogous calculation for other combinations of the helicity amplitudes leads to

$j^\mu_{RL} = 2E(0, -\cos\theta, i, \sin\theta)$
$j^\mu_{RR} = 2E(0, 0, 0, 0)$
$j^\mu_{LL} = 2E(0, 0, 0, 0)$
$j^\mu_{LR} = 2E(0, -\cos\theta, -i, \sin\theta)$.

Nonzero helicity combinations for the vector electron current (initial state)

$j^e_{RL} = 2E(0, -1, -i, 0)$
$j^e_{LR} = 2E(0, -1, i, 0)$

Finally, let us calculate the amplitude $M = \frac{e^2}{s} j^e \cdot j^\mu$ squared. For example $M_{RL\to RL}$ is formed by the product of currents j^e_{RL} and j^μ_{RL}, and has the form

$M_{RL\to RL} = \frac{e^2}{s}\, 2E(0, -1, -i, 0)\, 2E(0, -\cos\theta, i, \sin\theta)$

$= e^2(1 + \cos\theta)$

Summing up the four possible combinations for the initial state we find the amplitude squared

$$|M|^2 = |M_{RR}|^2 + |M_{LR}|^2 + |M_{RL}|^2 + |M_{LL}|^2$$
$$= e^4 \left(1 + \cos^2\theta\right)$$

so the cross section $\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}(1 + \cos^2\theta)$

integration over the solid angle Ω leads to the total cross section $\sigma = \frac{4\pi\alpha^2}{3s}$.

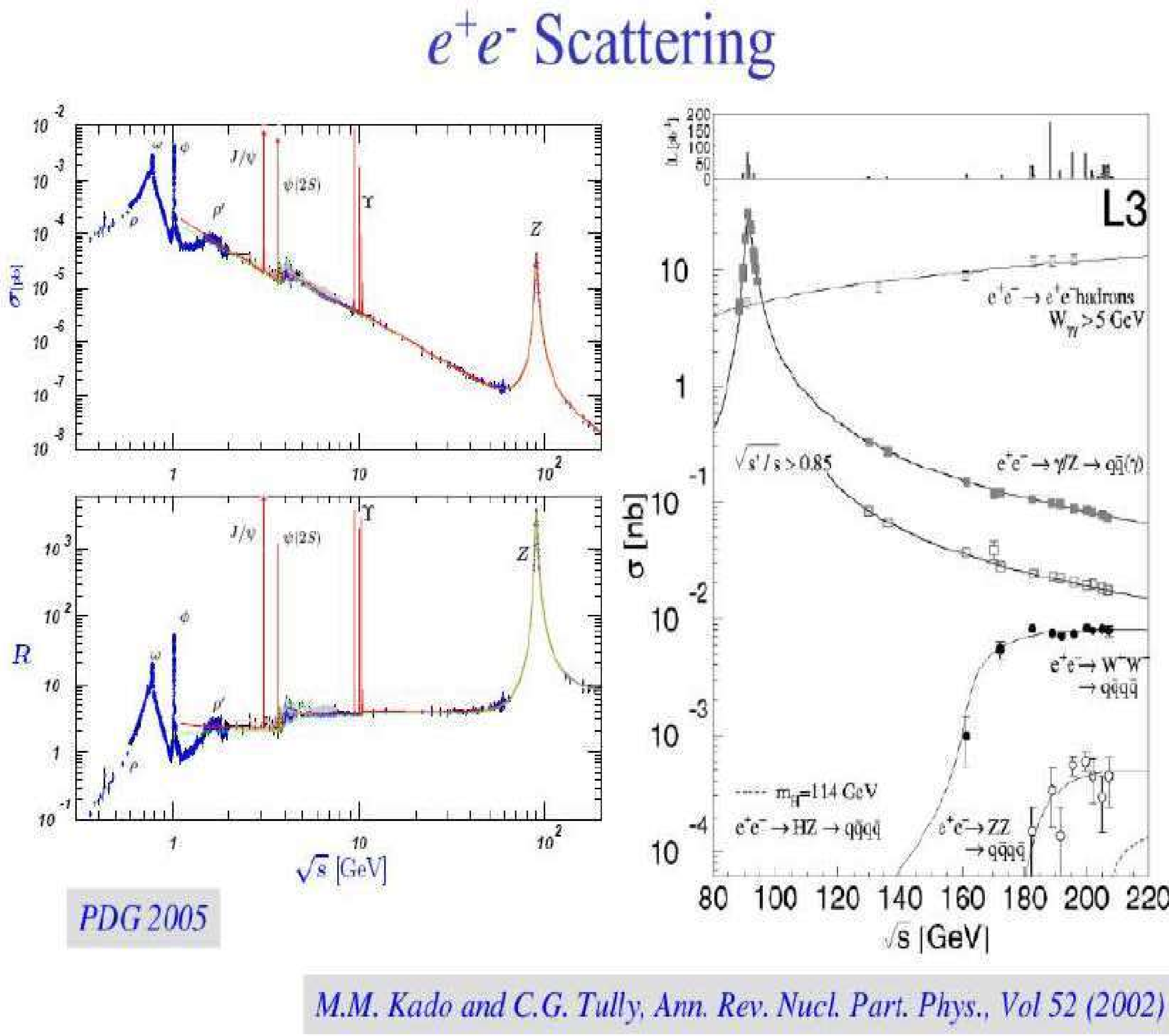

Рис. 1: **Experimental results for** σ **and** $R = \sigma(quarks)/sigma(\mu+\mu-)$

W-boson decay to the two fermions

Calculate the decay width of $W^- \to e^- \bar{\nu}_e$ using helicity amplitudes. Wave function W^-

$$W^\mu = \epsilon_i^\mu \, e^{-ipx}$$

where ϵ_i^μ, i =1,2,3 are the three polarization vectors of the W-boson (index μ =0,1,2,3). This function is the solution of the equation for a vector field in the form of plane waves. If the W boson moves along the z axis, three polarization vectors can be selected as

$$\epsilon_+^\mu = -\frac{1}{\sqrt{2}}(0,1,i,0),\ \epsilon_-^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0),$$
$$\epsilon_L^\mu = \frac{1}{m_W}(p_z,0,0,E)$$

where the transverse vectors $\epsilon_\pm$ correspond to circular polarization, ϵ_l corresponds to longitudinal polarization. The Feynman rule for the vertex $W^-(p_1) \to e^-(p_2)\bar{\nu}_e(p_3)$

$$\frac{g}{2\sqrt{2}}\, \gamma^\mu (1-\gamma^5)$$

g gauge constant $SU(2)$. Transition amplitude

$$M = \frac{g}{2\sqrt{2}}\epsilon_i^\mu(p_1)\bar{u}(p_2)\gamma_\mu(1-\gamma^5)u(p_3)$$

Let's write this amplitude as the product of the polarization vector × weak left current

$$M = \frac{g}{2\sqrt{2}}\epsilon_i^\mu(p_1)j_\mu, \text{ where}$$
$$j_\mu = \bar{u}(p_2)\gamma_\mu(1-\gamma^5)u(p_3)$$

and write down explicit representations for the 4-vectors p_1, p_2, p_3 in the rest system of the W-boson

$$p_1 = (m_W, 0, 0, 0),\ p_2 = (E_e, E_e\sin\theta, 0, E_e\cos\theta),$$
$$p_3 = (E_e, -E_e\sin\theta, 0, -E_e\cos\theta)$$

where the electron energy is $E_e = M_W/2$. An electron and an antineutrino fly apart along one straight line at an angle θ to the z axis. It is known that only the left helicity states of the particle and the right helical states of the antiparticle can contribute to the interactions of the left weak currents (see above), so it is possible to rewrite the left weak current in the form

$$j_\mu = \bar{u}_L(p_2)\gamma_\mu u_L(p_3)$$

where $\frac{1}{4}(1+\gamma_5)\gamma_\mu(1-\gamma^5) = \frac{1}{2}\gamma_\mu(1-\gamma_5)$.

For this current, we use the result obtained above relating to the muon current of the final state in the reaction $e^+e^- \to \mu^+\mu^-$, see the formula above, where you need to substitute $E = M_W/2$

$$j_\mu^{LR} = m_W(0, -\cos\theta, -i, \sin\theta)$$

Polarization vectors of the W-boson in its rest system

$$\epsilon_+^\mu = -\frac{1}{\sqrt{2}}(0, 1, i, 0),\ \epsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0),$$
$$\epsilon_L^\mu = \frac{1}{m_W}(0, 0, 0, 1)$$

Multiplying the 4-vectors ϵ_i^μ and j_μ^{LR}, we obtain the amplitudes for the three polarization states of the W boson

$$\begin{aligned}
M_- &= \frac{gm_W}{2\sqrt{2}}(0, 1, -i, 0) \cdot (0, -\cos\theta, -i, \sin\theta) \\
&= -\frac{g\,m_W}{2\sqrt{2}}(1 + \cos\theta), \\
M_+ &= -\frac{gm_W}{2\sqrt{2}}(0, 1, i, 0) \cdot (0, -\cos\theta, -i, \sin\theta) \\
&= \frac{g\,m_W}{2\sqrt{2}}(1 - \cos\theta), \\
M_L &= \frac{gm_W}{2}(0, 0, 0, 1) \cdot (0, -\cos\theta, -i, \sin\theta)
\end{aligned}$$

$$= \frac{g\, m_W}{2} \sin\theta$$

where the angular distributions come from

$$\begin{aligned} |M_-|^2 &= \frac{g^2\, m_W^2}{8}(1+\cos\theta)^2, \\ |M_+|^2 &= \frac{g^2\, m_W^2}{8}(1-\cos\theta)^2, \\ |M_L|^2 &= \frac{g^2\, m_W^2}{4} \sin^2\theta \end{aligned}$$

The angular distributions for the three polarization states W are different. The electron (left helicity) and the antineutrino (right) have a common spin of 1 in the direction of the antineutrino. The figure shows the distributions of $d\Gamma/d\cos\theta$

$$\frac{d\Gamma}{d\cos\theta} = -\frac{1}{\sin\theta}\frac{d\Gamma}{d\theta}$$

the distributions of $d\Gamma/d\theta$ differ by angle.

If the amplitude squared depends on $\cos\theta$, the angle distribution is changed by an additional factor $\sin\theta$, for an unpolarized W-boson, the distribution over the $\cos\theta$ of electron reconstructed in the rest system of the W boson is flat. For a transversely polarized W, the electron flies predominantly either by momentum or against it, but it cannot fly precisely at angles of 0 or 180 degrees due to the factor $\sin\theta$. For a longitudinally polarized W-boson, the electron flies out mainly at 90 degrees to the axis.

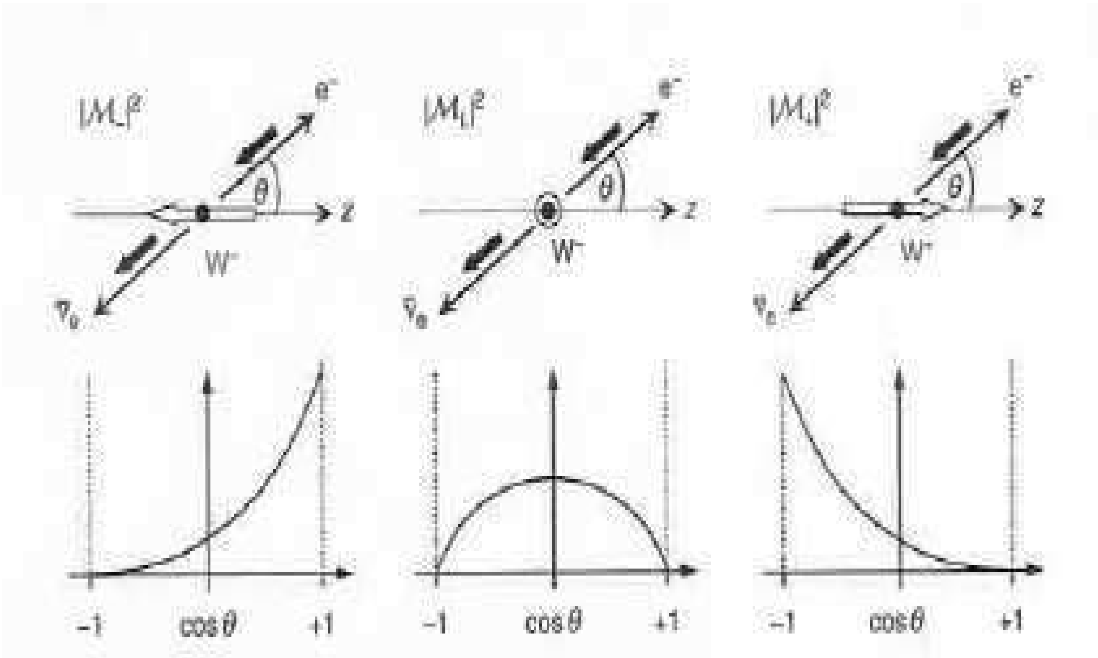

Рис. 2: **Angular distributions for three spin states of the W boson**

A complete matrix element squared taking into account the factor $\frac{1}{3}$ for averaging over polarization states

$$|M|^2 = \tfrac{1}{3}(|M_+|^2 + |M_-|^2 + |M_L|^2) = \tfrac{1}{3} g^2 m_W^2$$

The dependence on the angle dropped out. The angular distribution of $d\Gamma/d\cos\theta$ is flat. There is no preferred decay direction for the sum of the three polarization states. Decay width

$$\Gamma(W^- \to e^- \bar{\nu}_e) = \frac{1}{48\pi}\, g^2 m_W = \frac{G_F m_W^3}{6\pi\sqrt{2}}$$

where the relation $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$ is used.

Decay of the scalar particle (Higgs boson) into two b quarks

In this case, the amplitude has the form (spin 0)

$$M = \frac{m_b}{v}\, \bar{u}(p_2)u(p_3)$$

where m_b is the quark mass, v =246 GeV. Using previous formulae, we get spinors ($E = m_h/2$)

$$u_{1,2}(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},$$

$$u_{1,2}(p_3) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \sqrt{E}\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix},$$

from where, forming $u^+\gamma^0 u$, we can see that nonzero amplitudes can be

$$-M_{LL} = M_{RR} = \frac{m_b}{v} m_H$$

There is no angular dependence of the amplitude. The distribution of $\cos\theta$ is flat.