Appendix to Lecture 1 and Lecture 4 by A.I.Shevchenko
"Basics of particle physics"

M.N.Dubinin

(1) System of units $c = \hbar = 1$
(2) Symmetries of the Lagrangian, conserved currents (Lecture 4, slide 7, problem 1)
(3) $SU(2)$ symmetry, C,G-parity and resonance decays (Lecture 4, slide 11, problem 2)
Notations in Particle Data Group tables.

NUST MISiS

October 2021

System of units $c = \hbar = 1$

In this system of units the only nontrivial dimension is the dimension of mass M. Time and length have the dimension of $1/M$. This follows immediately from the definition of the Planck constant $E = \hbar\omega$, so for the dimensions which are denoted by square brackets one can write

$$[\hbar] = [\frac{ML^2}{T}], \quad [c] = [\frac{L}{t}], \tag{1}$$

so if $c = \hbar = 1$ then $[L] = [T] = [\frac{1}{M}]$. For the particle with mass M the value of $1/M$ is the *Compton wavelength*.

System of units $c = \hbar$ =1

1 GeV = 1.6·10^{-10} J = 1.6· 10^{-10} kg m^2 / s^2
(**Problem: Find in J the electron energy m =0.511 MeV in the potential $\triangle U$ 10^9 V**)
writing down $E = mc^2 \Rightarrow m$ at c=1
1 GeV = m · 9·10^{16}m^2/s^2 = 1.6·10^{-10}kg m^2/s^2
1 GeV/c^2 = 1.78·10^{-27}kg

then $E^2 = p^2 + m^2$ (not $E^2 = \vec{p}^{\,2}\ c^2 + m^2 c^4$)

Compton wavelength is measured in inverse GeV (in the following [dimension] is indicated in square brackets)
[m] $\lambda = \frac{\hbar}{mc} \Rightarrow \frac{1}{m}$, GeV^{-1}=0.197·$10^{-15}$m
= 0.197 fm, 1 fm = 10^{-15} m
Time is also measured in inverse GeV:
[s] $\frac{\lambda}{c} = \frac{\hbar}{mc^2} \Rightarrow \frac{1}{m}$, GeV^{-1}=6.582·$10^{-25}$ s

System of units $c = \hbar = 1$

Observable	kg, m, s	GeV, $\hbar$, c	$\hbar$=c=1
Time	s	$(\mathrm{GeV}/\hbar)^{-1}$	GeV^{-1}
Length	m	$(\mathrm{GeV}/\hbar\mathrm{c})^{-1}$	GeV^{-1}
Square	m^2	$(\mathrm{GeV}/\hbar\mathrm{c})^{-2}$	GeV^{-2}
Energy	kg m^2 s^{-2}	GeV	GeV
Momentum	kg m^2 s^{-1}	GeV/c	GeV
Mass	kg	GeV/c^2	GeV

Таблица: Units in the system c=$\hbar$=1 in comparison with SI. Transformation factors. Useful relation

$$\hbar c = 0.197 \text{ GeV} \times \text{fermi}$$

Lecture 4, slide 7. Lagrangian symmetries and conserved currents.

Action of a scalar field $\varphi(x)$ in a general form ($\partial_\mu = \partial/\partial x_\mu$)

$$S = \int \mathcal{L}\big(\varphi^I(x), \frac{\partial \varphi^I(x)}{\partial x_\mu}\big) d^4x \tag{2}$$

where the index I corresponds to a set of fields. In the following we use the notation $\frac{\partial \varphi^I(x)}{\partial x_\mu} = \varphi^I_{,\mu}$. Variation of action (1)

$$\delta S = \int \big[\frac{\partial \mathcal{L}}{\partial \varphi^I}\delta\varphi^I + \frac{\partial \mathcal{L}}{\partial \varphi^I_{,\mu}}\partial_\mu(\delta\varphi^I)\big] d^4x \tag{3}$$

sum over I is taken. Integrating the second term by parts and assuming that variations of fields are zero at the boundary of a space-time domain we get

$$\delta S = \int \big[\frac{\partial \mathcal{L}}{\partial \varphi^I}\delta\varphi^I - \partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^I_{,\mu}}\big](\delta\varphi^I) d^4x \tag{4}$$

so I equations of motion for the fields φ^I follow from the requirement $\delta S =$0

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi^I_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi^I} = 0. \tag{5}$$

Lecture 4, slide 7. Lagrangian symmetries and conserved currents.

Let us consider infinitesimal transformations of the general form

$$\varphi^I \Longrightarrow {\varphi'}^I = (\delta^{IJ} + \epsilon^a t_a^{IJ})\varphi^J \tag{6}$$

where ϵ^a are infinitesimal transformation parameters and t^{IJ} are some numerical parameters characterizing the transformation. Then variations of the field and the field derivative are

$$\delta\varphi^I = \epsilon^a t_a^{IJ}\varphi^J, \qquad \delta\varphi^I_{,\mu} = \epsilon^a t_a^{IJ}\partial_\mu\varphi^J \tag{7}$$

so variation of the Lagrangian when we shift the field by $\delta\varphi^I$ and the derivative of field by $\delta\varphi^I_{,\mu}$ looks as

$$\frac{\partial\mathcal{L}}{\partial\varphi^I}\delta\varphi^I + \frac{\partial\mathcal{L}}{\partial\varphi^I_{,\mu}} t_a^{IJ}\partial_\mu\varphi^J = 0 \tag{8}$$

and is equal to zero due to invariance of $\mathcal{L}$ with respect to transformations (5). Here we omitted the independent parameters ϵ^a. Using the field equation (4) we can write the derivative $\partial_\mu(\partial\mathcal{L}/\partial\varphi^I_{,\mu})$ instead of $\partial\mathcal{L}/\varphi^I$ in the first term of (7). After that sum of two terms forms the full derivative which expresses the conservation of current

$$J^a_\mu = \frac{\partial\mathcal{L}}{\partial\varphi^I_{,\mu}} t_a^{IJ}\partial_\mu\varphi^J, \quad \partial_\mu J^a_\mu = 0. \tag{9}$$

Lecture 4, slide 7. Lagrangian symmetries and conserved currents.

Problem 1 starts from the Lagrangian for complex scalar field

$$\mathcal{L} = \partial_\mu \varphi \partial_\mu \varphi^* - m^2 \varphi \varphi^* \tag{10}$$

where φ and φ^* are considered as independent fields, so φ^I, $I = 1, 2$ above denotes φ, φ^*. This Lagrangian is invariant with respect to

$$\varphi \Longrightarrow \varphi^{'} = e^{i\epsilon}\varphi, \quad \varphi^* \Longrightarrow \varphi^{'*} = e^{i\epsilon}\varphi^* \tag{11}$$

so the following calculation of the conserved current (7) is straightforward.

Problem 1a: Show that the charge Q defined as

$$Q = \int J_0 d^3x \tag{12}$$

is conserved, $\partial_0 Q = 0$. Calculate Q for the case of the solution of (3) which has the form $\varphi = Ne^{-i(kr-\omega t)}$ (plane wave, $\omega^2 = k^2 + m^2$), where N is the normalisation factor defined by normalisation of the energy of the field. Calculate N. Calculate the value of Q for this case.

Lecture 4, slide 11. $SU(2)$ symmetry and resonance decays. C-parity

(1) Mesons and baryons form isotopic multiplets which are described by irreducible representations of $SU(2)$ group

(2) Infinitesimal operators t_i of these representations, $[t_i, t_j] = i\epsilon_{ijk}t_k$ are called the isotopic spin operators. The operator $t^2 = t_1^2 + t_2^2 + t_3^2$, $t^2 = I(I+1)E$, E is unit operator, I is the isotopic spin of the multiplet. Electric charge of a particle - component of an isotopic multiplet - is expressed by means of t_3 and hypercharge Y

$$Q = t_3 + \frac{Y}{2} \tag{13}$$

which is known as Gell-Mann - Nishijima formula.

Examples

Proton and neutron form an isotopic doublet I =1/2 under the name nucleon. In terms of $SU(2)$ they are descibed by covariant isotopic spinor $\phi_\alpha = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ defined in the basis e^α ($e^\alpha e^\beta = \delta^{\alpha\beta}$)

$$e^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

basis vectors are transformed as $e^{\alpha'} = U_{\beta\alpha}e^\beta$, so $\varphi'_\alpha = U_{\alpha\beta}\varphi_\beta$ (fundamental representation, $SU(2)$ transformation is identical to group representation). We can write

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi(x), \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \psi(x), \quad or \quad N = \begin{pmatrix} p \\ n \end{pmatrix}$$

Lecture 4, slide 11. $SU(2)$ symmetry and resonance decays.

$\pi^{\pm}$ and π^{0} mesons form an isotopic triplet I =1 and are described by the second rank isotopic spinor transformed as $U \otimes U^{*}$, with one upper and one lower index. Such spinor is a sum of a spinor with zero trace and Kronecker term δ^a_b:

$$\varphi^a_b = (\varphi^a_b - \frac{1}{2}\delta^a_b\varphi^c_c) + \frac{1}{2}\delta^a_b\varphi^c_c,$$

which form two $SU(2)$ invariant supspaces. Zero trace basis looks as

$$e^1_2, \quad \frac{1}{\sqrt{2}}(e^1_1 - e^2_2), \quad e^2_1$$

and the one-dimensional space basis vector

$$\frac{1}{\sqrt{2}}(e^1_1 + e^2_2)$$

is ortogonal to the three basis vecor for zero-trace supspace. Matrix form looks as

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Lecture 4, slide 11. $SU(2)$ symmetry and resonance decays.

so for the triplet of π-mesons we have the form

$$\pi_b^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}$$

Such form is very convenient to define charge conjugation C, when

$$\pi^+ \to \pi^-, \quad \pi^- \to \pi^+, \quad \pi^0 \to \pi^0$$

which can be written in the matrix form

$$\pi_b^a \to \pi_a^b \quad or, equivalently \quad C\pi_b^a C^+ = \pi_a^b$$

Important $SU(2)$ transformation is the *charge symmetry transformation* where the matrix is

$$CS = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2$$

Matrix elements should be invariant under CS. Such transformation for the nucleon isotopic doublet

$$CS \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} n \\ -p \end{pmatrix}$$

Lecture 4, slide 11. $SU(2)$ symmetry and resonance decays.

CS transformation for the pion triplet

$$CS\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}CS^+ = \begin{pmatrix} -\frac{1}{\sqrt{2}}\pi^0 & -\pi^- \\ -\pi^+ & \frac{1}{\sqrt{2}}\pi^0 \end{pmatrix}$$

Important point is the relation between components of the isotopic spinor π^a_b and 3-vectors. Let us expand π^a_b in the basis of Pauli matrices

$$\pi^b_a = \frac{1}{\sqrt{2}}(\tau_i)^b_a V_i, \qquad (2A)$$

then it is well-known that $V_i = (V_1, V_2, V_3)$ transform as the 3-vector components which can be easily found:

$$V_i = \frac{1}{\sqrt{2}}(\tau_i)^a_b \varphi^b_a = \frac{1}{\sqrt{2}} Sp(\tau_i \varphi) \qquad (2B)$$

. Normalization condition

$$V_i V_i^* = (\varphi^+)^a_b \varphi^b_a = Sp(\varphi^+ \varphi).$$

Using these formulae a relation between e_+, e_0, e_- and the Cartesian basis e_x, e_y, e_z for 3-vectors V_i can be easily found

$$e_+ = \frac{1}{\sqrt{2}}(e_x + ie_y), e_0 = e_z, e_- = \frac{1}{\sqrt{2}}(e_x - ie_y)$$

.

Lecture 4, slide 11. $SU(2)$ symmetry and resonance decays.

To write $SU(2)$ invariant Lagrangian for nucleon-π interaction, compose the form $\overline{N}^a \pi^c_d N_b$ and take a sum over upper and lower pair of indices, so we have *line* $\times$ *matrix* $\times$ *column*, which is $SU(2)$ scalar. Then the Lagrangian($g_{\pi NN}\sqrt{2}$ - coupling constant)

$$L_{\pi NN} = g_{\pi NN}\sqrt{2}\,\overline{N}^a \pi^b_a N_b$$

or in terms of 3-vector V_i

$$L_{int} = g_{\pi NN}\,\overline{N}\tau_i \pi_i N$$

Substituting here matrix expressions we find an explicit form

$$L_{\pi NN} = g_{\pi NN}[\sqrt{2}\overline{p}n\pi^+ + \sqrt{2}\overline{n}p\pi^- + (\overline{p}p - \overline{n}n)\pi^0]$$

Less trivial example: decay of $f^0(1270)$ meson (T =0) to the two π-mesons (T=1), $f^0 \to \pi(p_1)\pi(p_2)$, where p_1 and p_2 are 4-momenta. $SU(2)$ invariant matrix element of this decay has the form

$$M(f^0 \to \pi\pi) = g_{f\pi\pi} f^0 \pi^a_b \pi^b_a = g f^0[\pi^+(p_1)\pi^-(p_2) + \pi^-(p_1)\pi^+(p_2) + \pi^0(p_1)\pi^0(p_2)]$$

so the probability of the channel $\pi^+\pi^-$ is higher by a factor of 2 than of the channel $\pi^0\pi^0$

$$\frac{BR(f^0 \to \pi^+\pi^-)}{BR(f^0 \to \pi^0\pi^0)} = 2$$

Problem 2. $SU(2)$ symmetry and resonance decays.

Note that $M(f^0 \to \pi\pi)$ is C-invariant only if the C-parity of the f^0 meson is equal to $+1$. Indeed, under C transformation

$$f^0 \to f^0, \quad \pi^a_b \pi^b_a \Longrightarrow \pi^b_a \pi^a_b \quad \textit{identical to} \quad \pi^a_b \pi^b_a$$

This matrix elemant respects also Bose statistics, since 2π are in D-state (f^0 has spatial spin 2 and P parity $+1$) so the spatial wave function (not written in the formulas above) is symmetric under $p_1 \to p_2$, $p_2 \to p_1$ and also the isotopic matrix element.

Brief denomination of Particle Data: $f^0(1270)\ I^G(J^{PC}) = 0^+ 2^{++}$.

For example, $\varphi(1020)$ meson, $I^G(J^{PC}) = 0^- 1^{--}$, decay to 2π is described by the matrix element of the same structure, as above. However, under C transformation $\varphi \to -\varphi$, so conservation of C invariance does not allow $\varphi \to \pi\pi$.

Problem 2, Lecture 4 Matrix element of $\rho(1450)$ which is isotopic triplet I =1, $I^G(J^{PC}) = 1^+ 1^{--}$, $C\rho^a_b C^+ = -\rho^b_a$ to $\pi\pi$ is described by $SU(2)$ invariant matrix element, invariant under C:

$$M(\rho \to \pi\pi) = g \rho^a_b \pi^b_c \pi^c_a$$

This matrix element defines relations between decay channels.

Problem 2. $SU(2)$ symmetry and resonance decays.

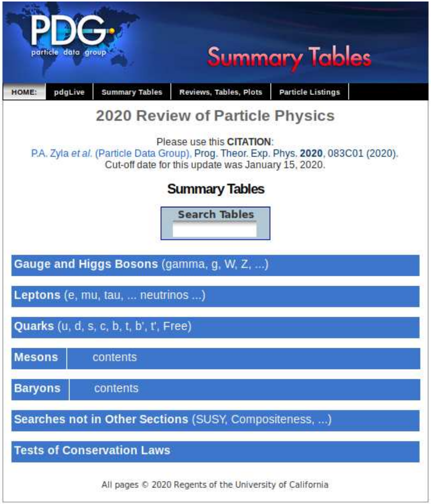

Рис.: see $https://pdg.lbl.gov/2020/tables/contents\_tables.html$

Problem 2. $SU(2)$ symmetry and resonance decays.

Citation: P.A. Zyla *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2020**, 083C01 (2020)

LIGHT UNFLAVORED MESONS
$(S = C = B = 0)$

For $I = 1$ (π, b, ρ, a): $u\bar{d}$, $(u\bar{u} - d\bar{d})/\sqrt{2}$, $d\bar{u}$;
for $I = 0$ $(\eta, \eta', h, h', \omega, \phi, f, f')$: $c_1(u\bar{u} + d\bar{d}) + c_2(s\bar{s})$

$\pi^\pm$

$I^G(J^P) = 1^-(0^-)$

Mass $m = 139.57039 \pm 0.00018$ MeV (S = 1.8)
Mean life $\tau = (2.6033 \pm 0.0005) \times 10^{-8}$ s (S = 1.2)
$c\tau = 7.8045$ m

π^0

$I^G(J^{PC}) = 1^-(0^{-+})$

Mass $m = 134.9768 \pm 0.0005$ MeV (S = 1.1)
$m_{\pi^\pm} - m_{\pi^0} = 4.5936 \pm 0.0005$ MeV
Mean life $\tau = (8.52 \pm 0.18) \times 10^{-17}$ s (S = 1.2)
$c\tau = 25.5$ nm

$f_2(1270)$

$I^G(J^{PC}) = 0^+(2^{++})$

Mass $m = 1275.5 \pm 0.8$ MeV
Full width $\Gamma = 186.7^{+2.2}_{-2.5}$ MeV (S = 1.4)

$f_2(1270)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
$\pi\pi$	$(84.2\ ^{+2.9}_{-0.9})$ %	S=1.1	623

PDG tables – G parity

After C transformation the neutral particles are transformed into themselves, their wave function is not changed (C parity +1) or changes sign (C parity -1). Charged particles are transformed to their antiparticles, so they do not have C-parity. We can generalize the notion of C-parity for (anti)particles from the same isotopic multiplet, combining C transformation and CS transformation, so $G = C \times CS$. For example, C-transformation for π triplet

$$\pi^- \to \pi^+, \quad \pi^+ \to \pi^-, \quad \pi^0 \to \pi^0$$

Under CS transformation described above

$$\pi^- \to -\pi^+, \quad \pi^+ \to -\pi^-, \quad \pi^0 \to -\pi^0$$

Combining C and CS, we have

$$\pi^- \to -\pi^-, \quad \pi^+ \to -\pi^+, \quad \pi^0 \to -\pi^0$$

so G-parity of the pion triplet is -1, or pions are eigenstates of the $C \times CS$ with the eigenvalue -1. G-parity conservation does not allow, for example, $\rho \to 3\pi$, $\phi \to 2\pi$ and so on. The value of G is indicated in the standard PDG denomination $I^G(J^{PC})$.