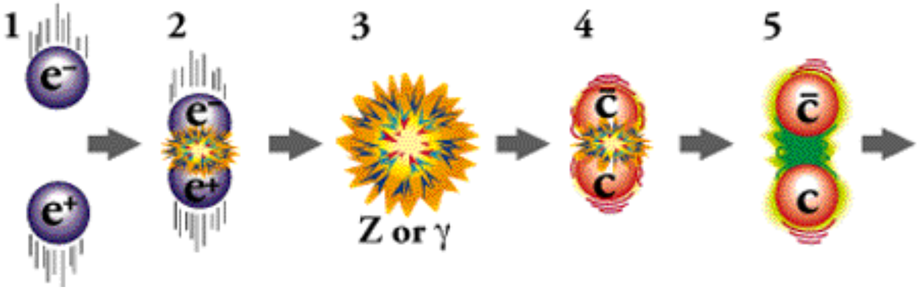

Electron-positron annihilation

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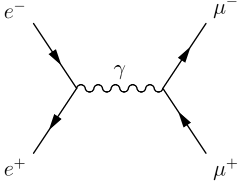
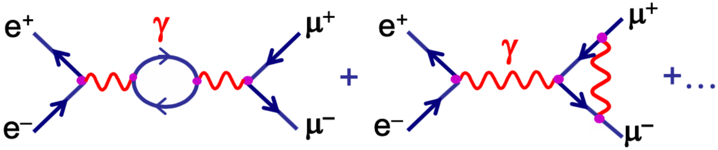
QED Calculations

How to calculate a cross section using QED (e.g. $e^+e^- \to \mu^+\mu^-$):

1. draw all possible Feynman Diagrams
   - for $e^+e^- \to \mu^+\mu^-$ there is just one lowest order diagram: $M \propto e^2 \propto \alpha_{em}$



   - plus many second order diagrams: $M \propto e^4 \propto \alpha_{em}^2$



2. for each diagram calculate the matrix element using Feynman rules

③ sum the individual matrix elements (i.e. sum the amplitudes):

$$M_{fi} = M_1 + M_2 + M_3 + \ldots$$

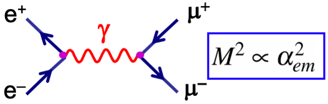
note: summing amplitudes $\Longrightarrow$ different diagrams can interfere either positively or negatively!

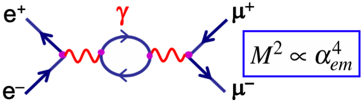
and then square

$$|M_{fi}|^2 = (M_1 + M_2 + M_3 + \ldots)(M_1^* + M_2^* + M_3^* + \ldots)$$

$\Longrightarrow$ this gives the full perturbation expansion in α_{em}

- for QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams:





4. calculate decay rate/cross section using previous formulae:
   - for a decay

$$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int \left|M_{fi}^2\right| \mathrm{d}\Omega \tag{1}$$

   - for scattering in the center-of-mass frame

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \tag{2}$$

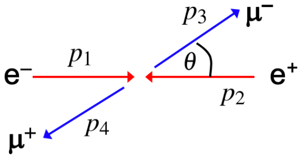
   - for scattering in lab. frame (neglecting mass of scattered particle)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \tag{3}$$

Electron-positron annihilation

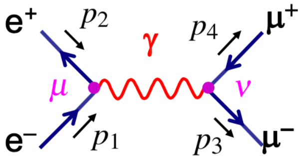
Consider the process: $e^+e^- \to \mu^+\mu^-$:

- work in C.o.M. frame (this is appropriate for most e^+e^- colliders):
  $p_1 = (E, 0, 0, p)$, $p_2 = (E, 0, 0, -p)$, $p_3 = (E, \vec{p}_f)$,
  $p_4 = (E, -\vec{p}_f)$



- only consider the lowest order Feynman diagram; from Feynman rules:

$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3)ie\gamma^\nu v(p_4)] \qquad (4)$$



- incoming anti-particle $\bar{v}$
- incoming particle u
- adjoint spinor written first

Electron-positron annihilation

- in the C.o.M. frame:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\vec{p}_f}{\vec{p}_i}|M_{fi}|^2 \text{ with } s = (p_1+p_2)^2 = (E+E)^2 = 4E^2 \quad (5)$$

- here $q^2 = (p_1+p_2)^2 = s$ and matrix element

$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)]\,\frac{-ig_{\mu\nu}}{q^2}\,[\bar{u}(p_3)ie\gamma^\nu v(p_4)] \quad (6)$$

becomes

$$M = -\frac{e^2}{s}g_{\mu\nu}\,[\bar{v}(p_2)\gamma^\mu u(p_1)]\,[\bar{u}(p_3)\gamma^\nu v(p_4)] \quad (7)$$

Electron and muon currents

- previously we introduced the four-vector current:

$$j^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi \tag{8}$$

which has same form as the two terms in [] in the matrix element of Eq. 7

- the matrix element can be written in terms of the e and μ currents:

$$(j_e)^{\mu} = \bar{v}(p_2)\gamma^{\mu}u(p_1) \text{ and } (j_{\mu})^{\nu} = \bar{u}(p_3)\gamma^{\nu}v(p_4) \tag{9}$$

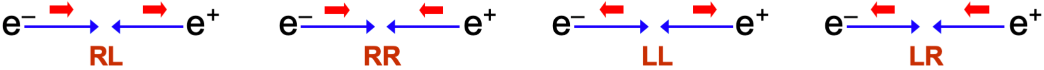
$$\Longrightarrow \; M = -\frac{e^2}{s}g_{\mu\nu}(j_e)^{\mu}(j_{\mu})^{\nu} \tag{10}$$

$$\boxed{M = \text{-}\frac{e^2}{s}j_e \cdot j_{\mu}} \tag{11}$$

- matrix element is a four-vector scalar product $\Longrightarrow$ Lorentz Invariant

Spin in e^+e^- annihilation

- in general, the electron and positron are not polarized, i.e. there is equal numbers of positive and negative helicity states
- there are four possible combinations of spins in the initial state:



- similarly there are four possible helicity combinations in the final state
- in total there are **16** combinations, e.g. **RL→RR**, **RL→RL**, ...

Spin in e^+e^- annihilation

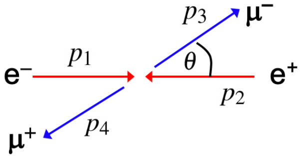
- to account for these states we need to sum over all **16** possible helicity combinations and then average over the number of initial helicity states:

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4}\left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \dots \right) \tag{12}$$

- i.e. need to evaluate $M = -\frac{e^2}{s} j_e \cdot j_\mu$ for all 16 helicity combinations
- fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements - this is an important feature of QED/QCD

Spin in e^+e^- annihilation

- in the C.o.M. frame in the limit $E \gg m$:
  $p_1 = (E, 0, 0, E)$, $p_2 = (E, 0, 0, -E)$,
  $p_3 = (E, E \sin\theta, 0, E\cos\theta)$,
  $p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$



  - left- and right-handed helicity spinors for particles and antiparticles:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix}; u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix}; v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}; v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix} \quad (13)$$

where $s = \sin\frac{\theta}{2}$, $c = \cos\frac{\theta}{2}$ and $N = \sqrt{E+m}$

Spin in e^+e^- annihilation

- in the limit $E \gg m$ these become:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; \quad v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; \quad v_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \tag{14}$$

where $s = \sin\frac{\theta}{2}$, $c = \cos\frac{\theta}{2}$

Spin in e^+e^- annihilation

- the initial-state e^- can either be in a left- or right-handed helicity state:

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}; \quad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}; \tag{15}$$

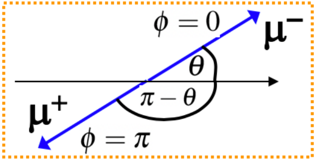
- for the initial state positron $(\theta = \pi)$ can have either:

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}; \quad v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}; \tag{16}$$

Spin in e^+e^- annihilation

- similarly for the final state μ^- with polar angle θ and choosing $\phi = 0$:

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}; \quad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}; \qquad (17)$$



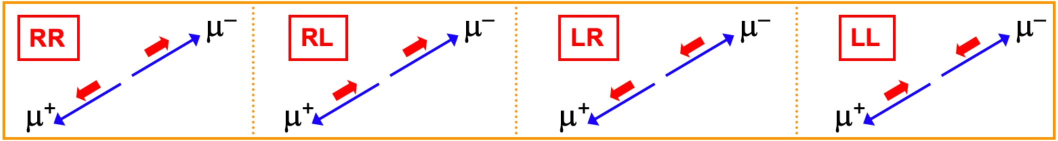
Spin in e^+e^- annihilation

- and for the final state μ^+ replacing $\theta \to \pi - \theta$, $\phi \to \pi$ obtain:

$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ s \end{pmatrix}; \quad v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; \tag{18}$$

using $\sin\left(\frac{\pi-\theta}{2}\right) = \cos\frac{\theta}{2}$, $\cos\left(\frac{\pi-\theta}{2}\right) = \sin\frac{\theta}{2}$, $e^{-i\pi} = -1$

- wish to calculate the matrix element $M = -\frac{e^2}{s} j_e \cdot j_\mu$
- first consider the muon current j_μ for 4 possible helicity combinations:



The muon current

- want to evaluate $(j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$ for all helicity combinations
- for arbitrary spinors ψ, ϕ it is straightforward to show that the components of $\overline{\psi}\gamma^\mu\phi$:

$$\overline{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \quad (19)$$

$$\overline{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \quad (20)$$

$$\overline{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \quad (21)$$

$$\overline{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2 \quad (22)$$

The muon current

- consider the $\mu^-_R \mu^+_L$ combination using $\psi = u_\uparrow$, $\phi = v_\downarrow$ with

$$v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix};\, u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}:$$

$$\overline{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0 \quad (23)$$

$$\overline{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\theta \quad (24)$$

$$\overline{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE(-c^2 - s^2 - c^2 - s^2) = 2iE \quad (25)$$

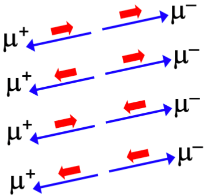
$$\overline{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta \quad (26)$$

The muon current

- hence the four-vector muon current for the **RL** combination is:

$$\overline{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta) \quad (27)$$

- the results for the four helicity combinations are:



$$\overline{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta) \quad \textbf{RL} \quad (28)$$
$$\overline{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4) = (0,0,0,0) \quad \textbf{RR} \quad (29)$$
$$\overline{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) = (0,0,0,0) \quad \textbf{LL} \quad (30)$$
$$\overline{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4) = 2E(0, -\cos\theta, -i, \sin\theta) \quad \textbf{LR} \quad (31)$$

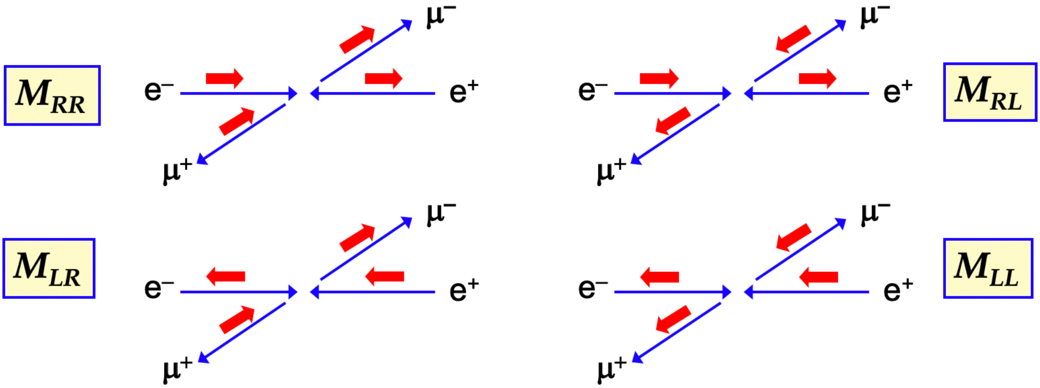
The muon current

In the limit $E \gg m$ only two helicity combinations are non-zero!

- this is an important feature of QED. It applies equally to QCD.
- in the Weak interaction only one helicity combination contributes.
- the origin of this will be discussed in the last part of this lecture
- but as a consequence of the 16 possible helicity combinations only four given non-zero matrix elements

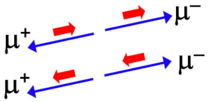
The muon current

- for $e^+e^- \to \mu^+\mu^-$ now only have to consider four matrix elements:



The muon current

- previously we derived the muon currents for the allowed helicities:



$$\mu_R^- \mu_L^+ : \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta) \quad (32)$$

$$\mu_L^- \mu_R^+ : \overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta) \quad (33)$$

- now need to consider the electron current

The electron current

- the incoming electron and positron spinors (L and R helicities) are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\; u_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix};\; v_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\; v_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}; \tag{34}$$

- the electron current can either be obtained from Eq. 19 or directly from the expressions for the muon current:

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \quad (j_\mu)^\mu = \overline{u}(p_3)\gamma^\mu v(p_4) \tag{35}$$

- taking the Hermitian conjugate of the muon current gives:

$$[\overline{u}(p_3)\gamma^\mu v(p_4)]^\dagger = \left[u(p_3)^\dagger\gamma^0\gamma^\mu v(p_4)\right]^\dagger \tag{36}$$

$$= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_3) \qquad (AB)^\dagger = B^\dagger A^\dagger \tag{37}$$

$$= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_3) \qquad \gamma^{0\dagger} = \gamma^0 \tag{38}$$

$$= v(p_4)^\dagger\gamma^0\gamma^\mu u(p_3) \qquad \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu \tag{39}$$

$$= \overline{v}(p_4)\gamma^\mu u(p_3) \tag{40}$$

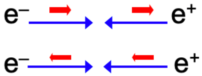
The electron current

- taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\overline{v}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_3) = [\overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4)]^* = 2E(0, -\cos\theta, -i, \sin\theta) \qquad (41)$$

$$\overline{v}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_3) = [\overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4)]^* = 2E(0, -\cos\theta, i, \sin\theta) \qquad (42)$$

To obtain the electron currents we simply need to set $\theta = 0$:

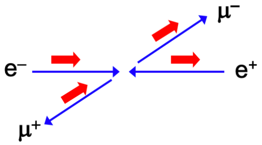


$$e_R^- e_L^+ : \quad \overline{v}_\downarrow(p_2)\gamma^\nu u_\uparrow(p_1) = 2E(0, -1, -i, 0) \qquad (43)$$

$$e_L^- e_R^+ : \quad \overline{v}_\uparrow(p_2)\gamma^\nu u_\downarrow(p_1) = 2E(0, -1, i, 0) \qquad (44)$$

Matrix element calculation

- we can now calculate $M = -\frac{e^2}{s} j_e \cdot j_\mu$ for the four possible helicity combinations
- e.g. we will do it for $e_R^- e_L^+ \to \mu_R^- \mu_L^+$ which we will denote M_{RR}:



here the first subscript refers to the helicity of the e^- and the second to the helicity of the μ^-. Don't need to specify other helicities due to "helicity conservation", only certain chiral combinations are non-zero

- using:

$$e_R^- e_L^+ : \quad (j_e)^\mu = \overline{v}_\downarrow(p_2)\gamma^\nu u_\uparrow(p_1) \qquad = 2E(0, -1, -i, 0) \qquad (45)$$

$$\mu_R^- \mu_L^+ : \quad (j_\mu)^\nu = \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) \qquad = 2E(0, -\cos\theta, i, \sin\theta) \qquad (46)$$

- gives:

$$M_{RR} = -\frac{e^2}{s}\left[2E(0, -1, -i, 0)\right] \cdot \left[2E(0, -\cos\theta, i, \sin\theta)\right] \qquad (47)$$

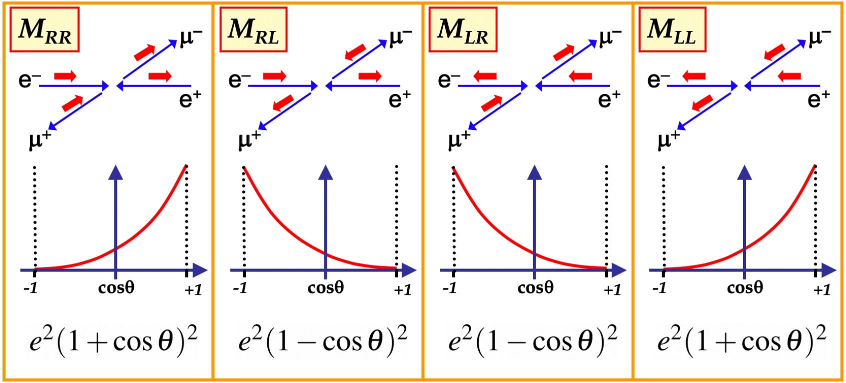
$$= -e^2(1 + \cos\theta) = -4\pi\alpha(1 + \cos\theta), \text{ where } \alpha = e^2/4\pi \approx 1/137 \qquad (48)$$

Matrix element calculation

Similarly:

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2 \quad (49)$$

$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2 \quad (50)$$



- assuming that the incoming electrons and positrons are **unpolarized**, all 4 possible initial helicity states are equally likely

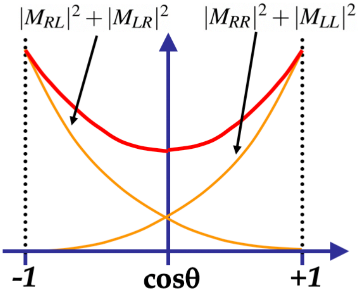
Differential cross section

- the cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2\right) \quad (51)$$

$$\frac{d\sigma}{d\Omega} = \frac{(4\pi\alpha)^2}{256\pi^2 s}\left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right) \quad (52)$$

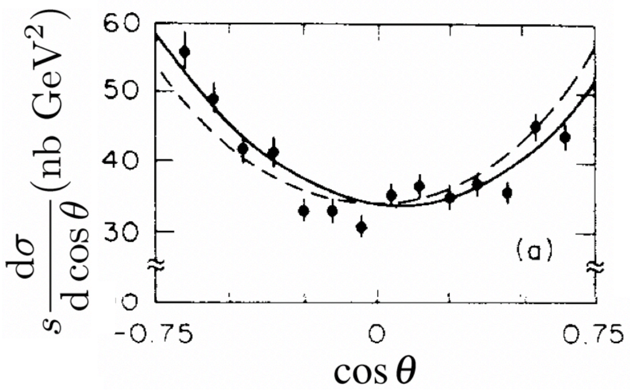
$$\Longrightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}(1+\cos^2\theta)} \quad (53)$$



Differential cross section: measurement

Example: $e^+e^- \to \mu^+\mu^-$ at $\sqrt{s} = 29$ GeV

Mark II Expt., M.E.Levi et al.,
Phys. Rev. Lett. 51 (1983) 1941



- - - - pure QED, $\mathcal{O}(\alpha^3)$
—— QED + Z contribution

Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

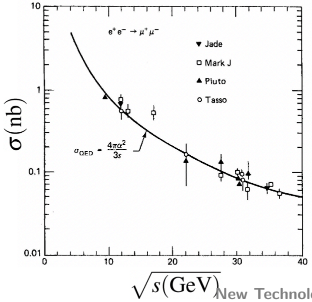
Total cross section: measurement

- the total cross section is obtained by integrating over θ, ϕ using:

$$\int (1+\cos^2\theta)\,d\Omega = 2\pi\int_{-1}^{+1}(1+\cos^2\theta)\,d\cos\theta = \frac{16\pi}{3} \quad (54)$$

giving the **QED** total cross section for the process $e^+e^- \to \mu^+\mu^-$

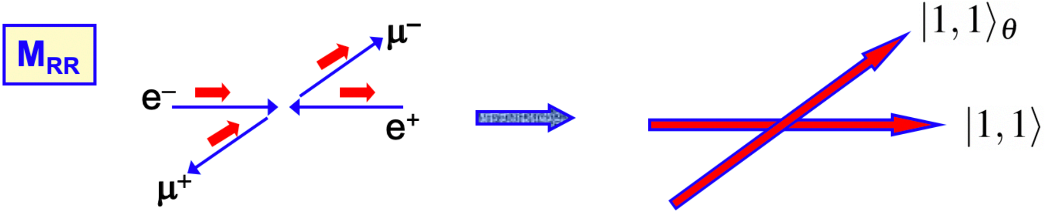
$$\boxed{\sigma = \frac{4\pi\alpha^2}{3s}} \quad (55)$$



- lowest order cross section calculation provides a good description of the data !
- this is an impressive result: from first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to 1%

Spin considerations ($E \gg m$)

- the angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum
- because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1, +1\rangle$ or $|1, -1\rangle$
- similarly, the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle θ, e.g.



- $\Longrightarrow$ $M_{RR} \propto \langle \psi | 1, 1 \rangle$ where ψ is the spin state, $|1, 1\rangle_\theta$ of the $\mu^+\mu^-$

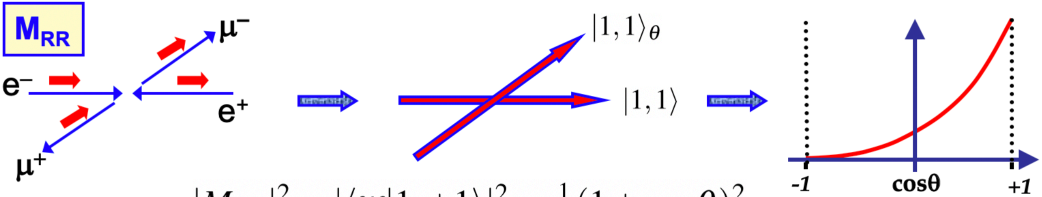
Spin considerations ($E \gg m$)

- to evaluate this need to express $|1,1\rangle_\theta$ in terms of eigenstates of S_z
- it is possible to show that:

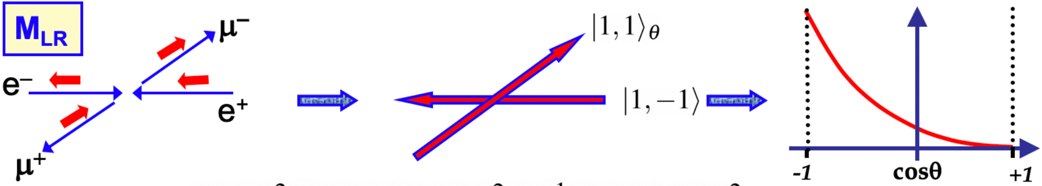
$$|1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle + \frac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \frac{1}{2}(1+\cos\theta)\,|1,+1\rangle \tag{56}$$

Spin considerations ($E \gg m$)

- using the wave-function for a spin 1 state along an axis at angle θ can immediately understand the angular dependence:



$$|M_{RR}|^2 \propto |\langle\psi|1,+1\rangle|^2 = \tfrac{1}{4}(1+\cos\theta)^2$$



$$|M_{LR}|^2 \propto |\langle\psi|1,-1\rangle|^2 = \tfrac{1}{4}(1-\cos\theta)^2$$

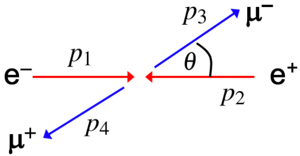
Lorentz Invariant form of Matrix Element

- note that the spin-averaged ME derived above is written in terms of the muon angle in the C.o.M. frame:

$$\langle |M_{fi}^2| \rangle = \frac{1}{4} \times \left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2 \right) \quad (57)$$

$$= \frac{1}{4} e^4 \left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2 \right) \quad (58)$$

$$= e^4 (1 + \cos^2\theta) \quad (59)$$



- the matrix element is Lorentz Invariant (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz-invariant 4-vector scalar products
- in the C.o.M. $p_1 = (E,0,0,E)$, $p_2 = (E,0,0,-E)$, $p_3 = (E, E\sin\theta, 0, E\cos\theta)$, $p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$ giving $p_1 \cdot p_2 = 2E^2$, $p_1 \cdot p_3 = E^2(1-\cos\theta)$, $p_1 \cdot p_4 = E^2(1+\cos\theta)$
- hence:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \left(\frac{t^2 + u^2}{s^2} \right) \quad (60)$$

Chirality

- the helicity eigenstates for a particle/anti-particle for $E \gg m$ are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; v_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \quad (61)$$

where $s = \sin\frac{\theta}{2}$, $c = \cos\frac{\theta}{2}$

- define the matrix:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}} \quad (62)$$

- in the limit $E \gg m$ the helicity states are also eigenstates of γ^5:

$$\gamma^5 u_\uparrow = +u_\uparrow; \quad \gamma^5 u_\downarrow = -u_\downarrow; \quad \gamma^5 v_\uparrow = -v_\uparrow; \quad \gamma^5 v_\downarrow = +v_\downarrow; \quad (63)$$

Chirality

- in general, define eigenstates of γ^5 as left- and right-handed chiral states: u_R, u_L, v_R, v_L, i.e.:

$$\gamma^5 u_R = +u_R; \quad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R; \quad \gamma^5 v_L = +v_L; \qquad (64)$$

- in the limit $E \gg m$ (and only in this limit):

$$u_R \equiv u_\uparrow; \quad u_L \equiv u_\downarrow; \quad v_R \equiv v_\uparrow; \quad v_L \equiv v_\downarrow; \qquad (65)$$

- this is a subtle but important point: in general the helicity and chiral eigenstates are not the same. It is only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates.
- chirality is an important concept in the structure of QED, and any interaction of the form $\overline{u}\gamma^\nu u$

Chirality

- in general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R; \quad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R; \quad \gamma^5 v_L = +v_L; \tag{66}$$

- define the projection operators:

$$\boxed{\mathrm{P}_R = \frac{1}{2}(1+\gamma^5); \quad P_L = \frac{1}{2}(1-\gamma^5)} \tag{67}$$

- the projection operators project out the chiral eigenstates:

$$P_R u_R = u_R; \quad P_R u_L = 0; \quad P_L u_R = 0; \quad P_L u_L = u_L; \tag{68}$$

$$P_R v_R = 0; \quad P_R v_L = v_L; \quad P_L v_R = v_R; \quad P_L v_L = 0 \tag{69}$$

- note P_R projects out right-handed particle states and left-handed anti-particle states
- we can then write any spinor in terms of it left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \frac{1}{2}(1+\gamma^5)\psi + \frac{1}{2}(1-\gamma^5)\psi \tag{70}$$

Chirality in QED

- in QED the basic interaction between a fermion and photon is:

$$ie\overline{\psi}\gamma^\mu\phi \tag{71}$$

- can decompose the spinors in terms of Left and Right-handed chiral components:

$$ie\overline{\psi}\gamma^\mu\phi = ie\left(\overline{\psi}_L + \overline{\psi}_R\right)\gamma^\mu(\phi_L + \phi_R) \tag{72}$$

$$= ie\left(\overline{\psi}_R\gamma^\mu\phi_R + \overline{\psi}_R\gamma^\mu\phi_L + \overline{\psi}_L\gamma^\mu\phi_R + \overline{\psi}_L\gamma^\mu\phi_L\right) \tag{73}$$

- using the properties of γ^5:

$$\boxed{\left(\gamma^5\right)^2 = 1; \quad \gamma^{5\dagger} = \gamma^5; \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5} \tag{74}$$

it is straightforward to show that

$$\overline{\psi}_R\gamma^\mu\phi_L = 0; \quad \overline{\psi}_L\gamma^\mu\phi_R = 0 \tag{75}$$

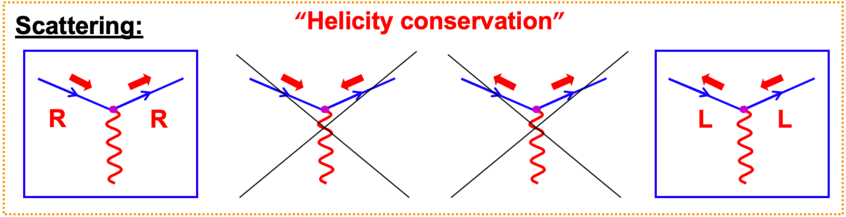
- hence only certain combinations of chiral eigenstates contribute to the interaction. This statement is always true

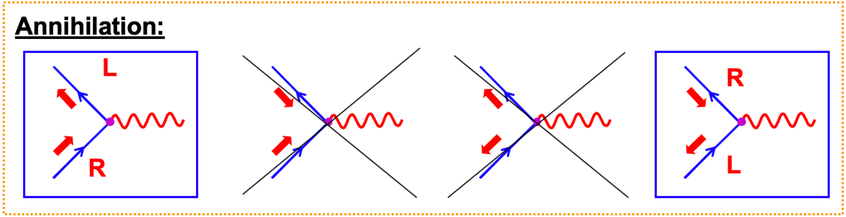
Chirality in QED

- for $E \gg m$ the chiral and helicity eigenstates are equivalent
- hence for $E \gg m$ only certain helicity combinations contribute to the QED vertex!
- this is why previously we found that for two of the four helicity combinations for the muon current were zero

Allowed QED Helicity Combinations

- in the ultra-relativistic limit the helicity eigenstates $\equiv$ chiral eigenstates
- in this limit, the only non-zero helicity combinations in QED are:





Summary

- in the center-of-mass frame the $e^+e^- \to \mu^+\mu^-$ differential cross section is:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta)} \quad (76)$$

  note: neglected masses of the muons, i.e. assumed $E \gg m_\mu$
- in QED only certain combinations of left- and right-handed chiral states give non-zero matrix elements
- chiral states defined by chiral projection operators:

$$\boxed{P_R = \frac{1}{2}(1 + \gamma^5); \quad P_L = \frac{1}{2}(1 - \gamma^5)} \quad (77)$$

Summary

- in limit $E \gg m$ the chiral eigenstates correspond to the helicity eigenstates and only certain helicity eigenstates give non-zero ME: