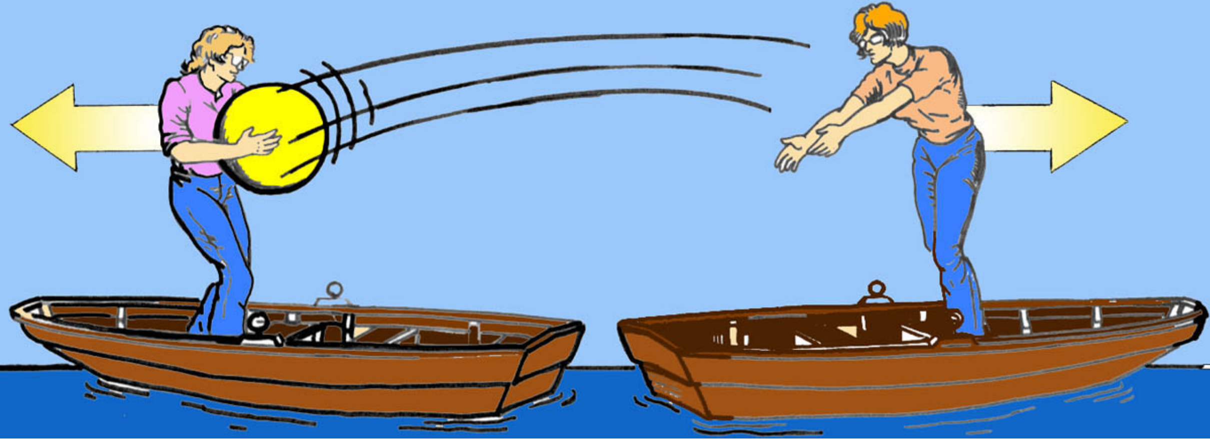


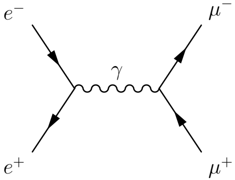
Particles and their interactions

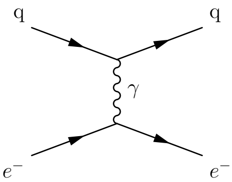
Lesya Shchutska

EPFL

The goals for the lecture

- we will work towards computation of the decay and scattering processes rates:
    - $e^+e^- \to \mu^+\mu^-$
    - $e^-q \to e^-q$





- we covered the relativistic calculation of particle decay rates and cross sections:

$$\sigma \propto \frac{|M|^2}{\text{flux}} \times (\text{phase space}) \quad (1)$$

- also went through the relativistic treatment of spin-half particles: **Dirac equation**
- here we will concentrate on the **Lorentz invariant Matrix Element**:
    - interaction by particle exchange
    - introduction to Feynman diagrams
    - the Feynman rules for QED

Interaction by particle exchange

- calculate transition rates from **Fermi's Golden rule**:

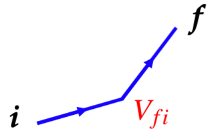
$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)} \tag{2}$$

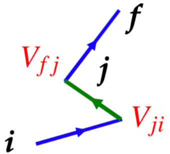
where T_{fi} is perturbation expansion for the Transition matrix element:

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle \ \langle j|V|i\rangle}{E_i - E_j} + \dots \tag{3}$$

- for particle scattering, the first two terms in the perturbation series can be viewed as:

"scattering in a potential"





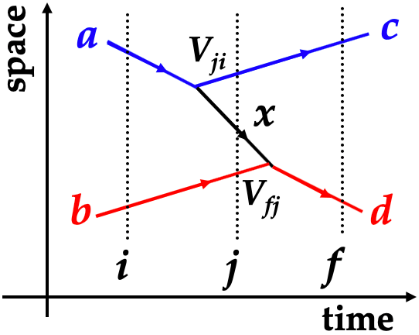
"scattering via an intermediate state"

Interaction by particle exchange

- **“Classical picture”**: particles act as sources for fields which give rise to a potential in which other particles scatter - **“action at a distance”**
- **“Quantum field theory picture”**: forces arise due to the exchange of virtual particles. Action at a distance does not exits, and forces between particles are due to particles

Interaction by particle exchange

- consider the particle interaction $a + b \to c + d$ which occurs via an intermediate state
- one possible space-time picture of this process is:



- initial state i: $a + b$
- final state f: $c + d$
- intermediate state j: $c + b + x$
- this time-ordered diagram corresponds to a "emitting" x and then b absorbing x

Interaction by particle exchange

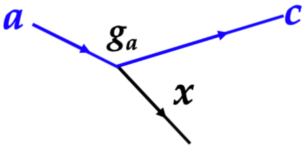
- the corresponding term in the perturbation expansion is:

$$T_{fi} = \frac{\langle f|V|j\rangle \langle j|V|i\rangle}{E_i - E_j} \tag{4}$$

$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle \langle c+x|V|a\rangle}{(E_a + E_b) - (E_c + E_x + E_b)} \tag{5}$$

- T_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it

Interaction by particle exchange



- need an expression for $\langle c + x | V | a \rangle$ in non-invariant matrix element T_{fi}
- ultimately aiming to obtain Lorentz invariant ME
  - recall T_{fi} is related to the invariant matrix element by

$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi} \tag{6}$$

    where k runs over all particles in the matrix element
  - here we have:

$$\langle c + x | V | a \rangle = \frac{M_{(a \to c+x)}}{(2E_a 2E_c 2E_x)^{1/2}} \tag{7}$$

    $M_{(a \to c+x)}$ is the "Lorentz invariant" matrix element for $a \to c + x$

Interaction by particle exchange

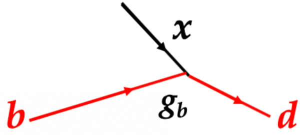
- the simplest Lorentz invariant quantity is a scalar, in this case:

$$\langle c + x | V | a \rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}} \tag{8}$$

  g_a is a measure of the strength of the interaction $a \rightarrow c + x$
  - the matrix element is only LI in the sense that it is defined in terms of LI wave-function normalizations and that the form of the couplings is LI
  - in this "illustrative" example g is not dimensionless

- similarly

$$\langle d | V | x + b \rangle = \frac{g_b}{(2E_b 2E_d 2E_x)^{1/2}} \tag{9}$$



- giving

$$T_{fi}^{ab} = \frac{\langle d | V | x + b \rangle \langle c + x | V | a \rangle}{(E_a + E_b) - (E_c + E_x + E_b)} = \frac{1}{2E_x} \cdot \frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)} \tag{10}$$

Interaction by particle exchange

- the "Lorentz invariant" matrix element for the entire process is

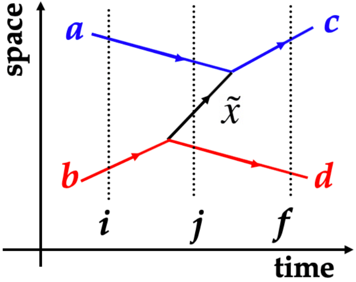
$$M_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)} \quad (11)$$

Here:

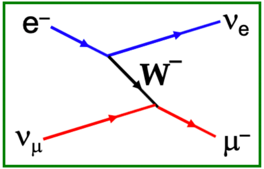
- M_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it $\Longrightarrow$ it is **not** Lorentz invariant, since order of event in time depends on frame
- momentum is conserved at each interaction vertex but not energy: $E_j \neq E_i$
- particle x is "on-mass shell" (or "on-shell"), i.e. $E_x^2 = \vec{p}_x^{\,2} + m^2$

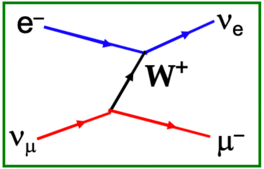
Interaction by particle exchange

- need to consider also the other time ordering for the process



- this time-ordered diagram corresponds to b "emitting" $\tilde{x}$ and then a absorbing $\tilde{x}$
- $\tilde{x}$ is the anti-particle of x, e.g.:





- the Lorentz-invariant matrix element for this time ordering is:

$$M_{fi}^{ba} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_b - E_d - E_x)} \qquad (12)$$

Interaction by particle exchange

- in QM need to sum over ME corresponding to same final state:

$$M_{fi} = M_{fi}^{ab} + M_{fi}^{ba} = \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x} \right) \tag{13}$$

$$= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x} \right), \text{ since } E_a + E_b = E_c + E_d \tag{14}$$

- this leads to:

$$M_{fi} = \frac{g_a g_b}{2E_x} \cdot \frac{2E_x}{(E_a - E_c)^2 - E_x^2} = \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2} \tag{15}$$

Interaction by particle exchange

- from the first time ordering:

$$E_x^2 = \vec{p}_x^2 + m_x^2 = (\vec{p}_a - \vec{p}_c)^2 + m_x^2, \tag{16}$$

giving

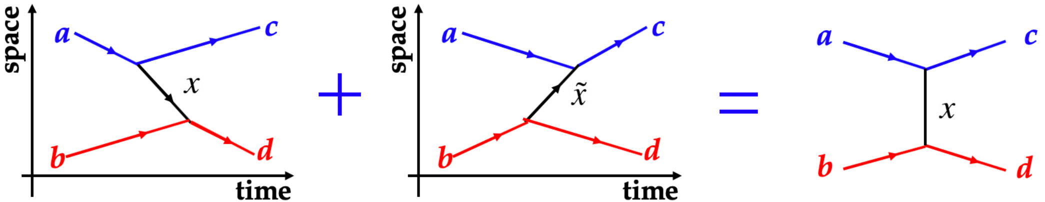
$$M_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2} = \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2} \tag{17}$$

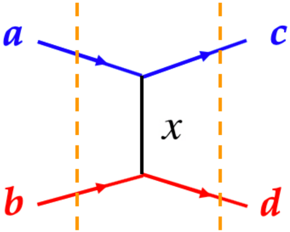
$$\Longrightarrow \quad M_{fi} = \frac{g_a g_b}{q^2 - m^2}, q = p_a - p_c \tag{18}$$

- after summing over all possible time orderings, M_{fi} is Lorentz invariant, i.e. the sum over all time orderings gives a frame independent matrix element
- exactly the same result would have been obtained by considering the annihilation process

Feynman diagrams

- the sum over all possible time-orderings is represented by a Feynman diagram





In a Feynman diagram:

- the LHS represents the initial state
- the RHS is the final state
- everything in between is "how the interaction happened"

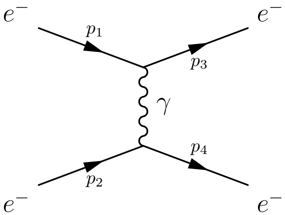
Feynman diagrams

- it is important to remember that **energy and momentum** are conserved at each interaction vertex in the diagram
- the factor $1/\left(q^2 - m_x^2\right)$ is the propagator; it arises naturally from the above discussion of interaction by particle exchange

The matrix element $M_{fi} = \dfrac{g_a g_b}{q^2 - m^2}$ depends on:

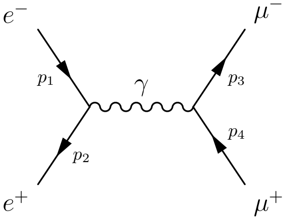
- the fundamental strength of the interaction at the two vertices g_a, g_b
- the four-momentum, q, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices.
- note, q^2 can be either positive or negative

Feynman diagrams



"t-channel"

- here $q = p_1 - p_3 = p_4 - p_2 = t$
- for **elastic scattering**: $p_1 = (E, \vec{p}_1)$; $p_3 = (E, \vec{p}_3)$, $q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2 < 0$
- $q^2 < 0$: is called **"space-like"**

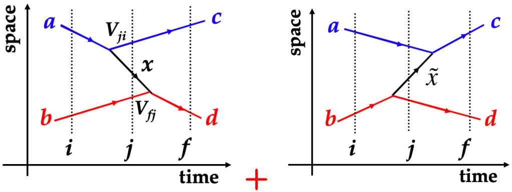


"s-channel"

- here $q = p_1 + p_2 = p_3 + p_4 = s$
- for **in CoM**: $p_1 = (E, \vec{p})$; $p_2 = (E, -\vec{p})$, $q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2 > 0$
- $q^2 > 0$: is called **"time-like"**

Virtual particles

"Time-ordered QM"

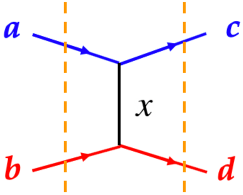


=

- momentum conserved at vertices
- energy **not** conserved at vertices
- exchanged particle "on mass shell"

$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

"Feynman diagram"



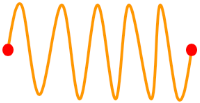
$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- momentum **and** energy conserved at interaction vertices
- exchanged particle "off mass shell": **virtual particle**

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

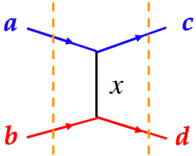
Virtual particles

Can think of observable "on mass shell" particles as propagating waves and unobservable virtual particles as normal modes between the sources particles:



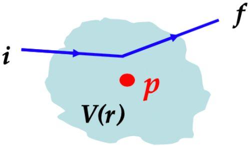
$V(r)$ from particle exchange

- can view the scattering of an electron by a proton at rest in two ways:
  - interaction by particle exchange in 2^{nd} order perturbation theory:



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

  - could also evaluate the same process in first order perturbation theory treating proton as a fixed source of a field which gives a rise to a potential $V(r)$



$$M = \langle \Psi_f | V | \Psi_i \rangle$$

we will obtain the same expression for M_{fi}

using $V(r) = g_a g_b \frac{e^{-mr}}{r}$ - **Yukawa potential**

- in this way can relate potential and forces to the particle exchange picture
- however, scattering from a fixed potential $V(r)$ is not a relativistic invariant view

Quantum electrodynamics - QED

- now consider the interaction of an electron and τ lepton by the exchange of a photon
- the general ideas still hold but also need to account for the spin of the electron/τ lepton and also the spin(polarization) of the virtual photon
- the basic interaction between a photon and a charged particle can be introduced by making the minimal substitution:

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi \quad (q \text{ is a charge}) \tag{19}$$

in QM:

$$\vec{p} = -i\vec{\nabla}; \quad E = i\frac{\partial}{\partial t} \tag{20}$$

Therefore make substitution: $i\partial_\mu \to i\partial_\mu - qA_\mu$,
where $A_\mu = \left(\phi, -\vec{A}\right); \quad \partial_\mu = \left(\frac{\partial}{\partial t}, +\vec{\nabla}\right)$

Quantum electrodynamics - QED

- the Dirac equation:

$$\gamma^{\mu}\partial_{\mu}\Psi + im\Psi = 0 \implies \gamma^{\mu}\partial_{\mu}\Psi + iq\gamma^{\mu}A_{\mu}\Psi + im\Psi = 0 \tag{21}$$

$$(\times i) \implies i\gamma^{0}\frac{\partial\Psi}{\partial t} + i\vec{\gamma}\vec{\boldsymbol{\nabla}}\Psi - q\gamma^{\mu}A_{\mu}\Psi - m\Psi = 0 \tag{22}$$

$$i\gamma^{0}\frac{\partial\Psi}{\partial t} = \gamma^{0}\widehat{H}\psi = m\Psi - i\vec{\gamma}\vec{\boldsymbol{\nabla}}\Psi + q\gamma^{\mu}A_{\mu}\Psi \tag{23}$$

$$\times\gamma^{0}: \quad \widehat{H}\Psi = \underbrace{\left(\gamma^{0}m - i\gamma^{0}\vec{\gamma}\vec{\boldsymbol{\nabla}}\right)\Psi}_{\text{combined rest mass + K.E.}} + \underbrace{q\gamma^{0}\gamma^{\mu}A_{\mu}\Psi}_{\text{potential energy}} \tag{24}$$

- we can identify the potential energy of a charged spin-half particle in an electromagnetic field as:

$$\boxed{\widehat{V}_{D} = q\gamma^{0}\gamma^{\mu}A_{\mu}} \tag{25}$$

(note the A_0 term is just $q\gamma^{0}\gamma^{0}A_{0} = q\phi$)

Quantum electrodynamics - QED

- the final complication is that we have to account for the photon polarization states:

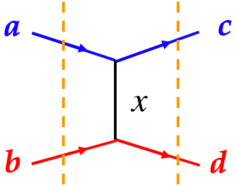
$$A_\mu = \epsilon_\mu^{(\lambda)} e^{i(\vec{p}\cdot\vec{r} - Et)} \tag{26}$$

e.g. for a real photon propagating in the z direction we have two orthogonal transverse polarization states:

$$\epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{27}$$

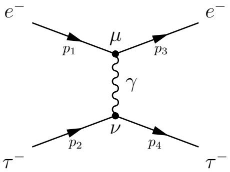
it's possible to also choose circularly polarized states

Quantum electrodynamics - QED



- previously with the example of a simple spin-less interaction we had:

$$M = \underbrace{\langle \Psi_c | V | \Psi_a \rangle}_{=g_a} \frac{1}{q^2 - m_x^2} \underbrace{\langle \Psi_d | V | \Psi_b \rangle}_{=g_b} \quad (28)$$



- in QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expressions for $\widehat{V}_D$. At the end, after summing over all the photon polarizations, we obtain:

$$M = \underbrace{\left[u_e^\dagger(p_3) q_e \gamma^0 \gamma^\mu u_e(p_1) \right]}_{\text{interaction of } e^- \text{ with photon}} \underbrace{\sum_\lambda \frac{\epsilon_\mu^\lambda \left(\epsilon_\nu^\lambda\right)^*}{q^2}}_{\text{massless } \gamma, \text{ sum over polarizations}} \underbrace{\left[u_\tau^\dagger(p_4) q_\tau \gamma^0 \gamma^\nu u_\tau(p_2) \right]}_{\text{interaction of } \tau^- \text{ with photon}} \quad (29)$$

All the physics of QED is in the above equation!

Quantum electrodynamics - QED

- the sum over the polarizations of the **virtual** photon has to include longitudinal and scalar contributions, i.e. 4 polarization states:

$$\epsilon^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \epsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{30}$$

  and it is equal to:

$$\sum_{\lambda} \epsilon_{\mu}^{\lambda} \left(\epsilon_{\nu}^{\lambda}\right)^{*} = -g_{\mu\nu} \tag{31}$$

  (at this point we take this statement as is)

- the invariant matrix element becomes:

$$M = \left[u_e^{\dagger}(p_3) q_e \gamma^0 \gamma^{\mu} u_e(p_1)\right] \frac{-g_{\mu\nu}}{q^2} \left[u_{\tau}^{\dagger}(p_4) q_{\tau} \gamma^0 \gamma^{\mu} u_{\tau}(p_2)\right] \tag{32}$$

Quantum electrodynamics - QED

- using the definition of the adjoint spinor $\overline{\Psi} = \Psi^\dagger \gamma^0$:

$$\boxed{\mathrm{M} = [\bar{u}_e(p_3) q_e \gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\bar{u}_\tau(p_4) q_\tau \gamma^\mu u_\tau(p_2)]} \tag{33}$$

- this is a remarkably simple expression!
- $\bar{u}_1 \gamma^\mu u_2$ transforms as a four vector
- writing

$$j_e^\mu = \bar{u}_e(p_3)\gamma^\mu u_e(p_1), \quad j_\tau^\nu = \bar{u}_\tau(p_4)\gamma^\nu u_\tau(p_2) \tag{34}$$

we get that M is Lorentz invariant:

$$M = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2} \tag{35}$$

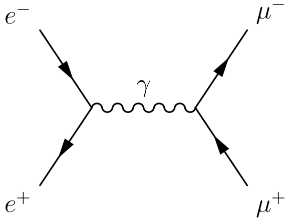
Feynman rules for QED

- it should be remembered that the expression

$$M = [\bar{u}_e(p_3) q_e \gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\bar{u}_\tau(p_4) q_\tau \gamma^\nu u_\tau(p_2)] \quad (36)$$

hides a lot of complexity:
  - we have summed over all possible time-orderings and summed over all polarization states of the virtual photon
  - if we are presented with a new Feynman diagram we do not want to go through the full calculation again
  - fortunately, this is not necessary: can just write down matrix element using a set of simple rules

Quantum electrodynamics - QED



Basic Feynman rules

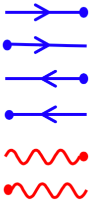
- propagator factor for each internal line: i.e. each internal virtual particle
- Dirac spinor for each external line, i.e. each real incoming or outgoing particle
- vertex factor for each vertex

Basic rules for QED

- external lines:

spin 1/2
- **incoming particle** $u(p)$
- **outgoing particle** $\overline{u}(p)$
- **incoming antiparticle** $\overline{v}(p)$
- **outgoing antiparticle** $v(p)$

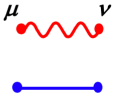
spin 1
- **incoming photon** $\varepsilon^{\mu}(p)$
- **outgoing photon** $\varepsilon^{\mu}(p)^{*}$



- internal lines (propagators):

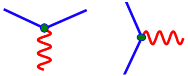
spin 1 **photon** $-\dfrac{ig_{\mu\nu}}{q^2}$

spin 1/2 **fermion** $\dfrac{i(\gamma^{\mu}q_{\mu}+m)}{q^2-m^2}$



- vertex factors:

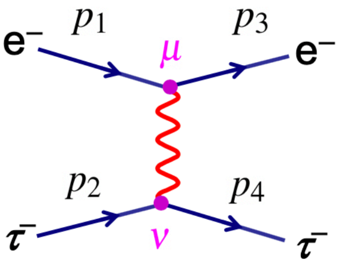
spin 1/2 **fermion (charge -|e|)** $ie\gamma^{\mu}$

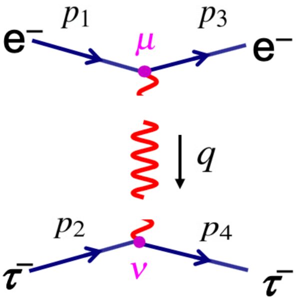


- matrix element $-iM$ = product of all factors

Examples

1





$$\bar{u}_e(p_3)\,[ie\gamma^\mu]\,u_e(p_1)$$

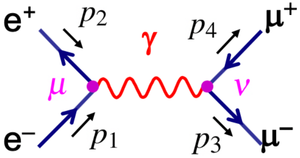
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\bar{u}_\tau(p_4)\,[ie\gamma^\nu]\,u_\tau(p_2)$$

$$-iM = [\bar{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\,\frac{-ig_{\mu\nu}}{q^2}\,[\bar{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

which is the same expression as we obtained previously

2



$$-iM = [\bar{v}(p_2) ie\gamma^{\mu} u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3) ie\gamma^{\nu} v(p_4)]$$

- at each vertex the adjoint spinor is written first
- each vertex has a different index
- the $g_{\mu\nu}$ of the propagator connects the indices at the vertices

Summary

- interaction by particle exchange naturally gives rise to Lorentz Invariant Matrix Element of the form

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2} \tag{37}$$

- derived the basic interaction in QED taking into account the spins of the fermions and polarization of the virtual photons:

$$-iM = [\bar{u}(p_3) ie\gamma^{\mu} u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_4) ie\gamma^{\nu} u(p_2)] \tag{38}$$

- we now have all the elements to perform proper calculations in QED!