Particles and their interactions

Lesya Shchutska

EPFL

Quantum mechanics (reminder)

Wave function:

- quantum mechanics (QM) takes into account the wave-particle duality, and implies that one can never predict the exact state of a particle position and momentum, with certainty
- one can thus no longer represent the state of the particle by a vector position known with unlimited precision at each time t. Instead, the state of the particle is represented by a wave function:

$$\vec{r}(t) \implies \Psi(\vec{r}, t) \text{ or } \Psi(\vec{p}, E) \qquad (1)$$

Meaning of the wave function:

- the concept of precise trajectory is replaced by a probability density to find the particle at a given position at a given time:

$$\rho(\vec{r}, t) = |\Psi(\vec{r}, t)|^2 = \Psi^*(\vec{r}, t)\Psi^(\vec{r}, t) \qquad (2)$$

$$\text{probability} \equiv \rho \text{d} r^3 \qquad (3)$$

Quantum mechanics (reminder)

Observables:

- any measurable physical quantity A can be associated with a linear operator $\widehat{A}$ such that if one knows $\Psi(x)$ the expected value of that quantity can be obtained through:

$$\langle A \rangle = \int \Psi^*(x)\widehat{A}\Psi(x)\mathrm{d}x \tag{4}$$

- for position, momentum (in 1D) and energy the corresponding operators and the expectation values are:

$$x \implies \widehat{X} = x \implies \langle x \rangle = \int \Psi^*(x,t)x\Psi(x,t)\mathrm{d}x \tag{5}$$

$$p_x \implies \widehat{P_x} = -i\hbar\frac{\partial}{\partial x} \implies \langle p_x \rangle = \int \Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\mathrm{d}x \tag{6}$$

$$E \implies \widehat{E} = i\hbar\frac{\partial}{\partial t} \implies \langle E \rangle = \int \Psi^*(x,t)\left(i\hbar\frac{\partial}{\partial t}\right)\Psi(x,t)\mathrm{d}x \tag{7}$$

Quantum mechanics (reminder)

Heisenberg's uncertainty principle:

- for two physical quantities to be simultaneously measured their operators should be commutative, i.e.,

$$\left[\widehat{A}, \widehat{B}\right] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A} = 0 \tag{8}$$

- examples:

$$\left[\widehat{X}, \widehat{Y}\right] = \left[\widehat{Y}, \widehat{Z}\right] = \left[\widehat{Z}, \widehat{X}\right] = 0, \tag{9}$$

$$\left[\widehat{P_x}, \widehat{P_y}\right] = \left[\widehat{P_y}, \widehat{P_z}\right] = \left[\widehat{P_z}, \widehat{P_x}\right] = 0, \tag{10}$$

$$\left[\widehat{X}, \widehat{P_y}\right] = \left[\widehat{X}, \widehat{P_z}\right] = 0 \tag{11}$$

$$\left[\widehat{X}, \widehat{P_x}\right] = i\hbar, \left[\widehat{Y}, \widehat{P_y}\right] = i\hbar, \left[\widehat{Z}, \widehat{P_z}\right] = i\hbar \tag{12}$$

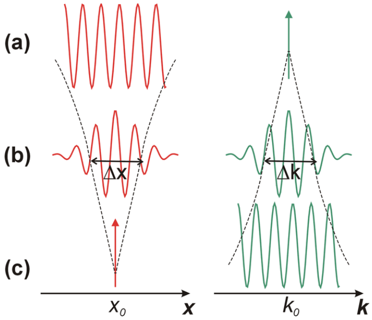
Quantum mechanics (reminder)

- the last Eq. 12 leads to Heisenberg's uncertainty principle:

$$\boxed{\sigma_x \sigma_{p_x} > \frac{\hbar}{2}} \tag{13}$$

- it arises due to the matter-wave nature of all quantum objects:

- (a) a pure wave of fixed frequency has no spatial localization but p is well defined as $p \propto k \propto 1/\lambda \propto \nu$
- (b) a wave packet with spatial dispersion Δx and frequency dispersion Δk, due to the link between the two particle representations, the spatial dispersion is inversely proportional to the frequency dispersion $\sigma_x \sigma_{p_x} > \frac{\hbar}{2}$
- (c) a pure particle is localized but has no determined frequency



Wave-Particle duality

The Schrödinger equation (1926)

- the first QM equation was established by Schrödinger for non-relativistic particles. He assumed that the solution should be of the same form as for the electromagnetic wave:

$$\Psi = Ne^{i(kx-\omega t)} \text{ or } \Psi = Ne^{i(px-Et)/\hbar} (\text{as } E = \hbar\omega, p = \hbar k) \tag{14}$$

- start with the non-relativistic relation between energy and momentum:

$$E = \frac{p^2}{2m} + V| \times \Psi \implies E\Psi = \frac{p^2}{2m}\Psi + V\Psi \tag{15}$$

- take the derivative of the wave function Ψ:

$$\frac{\partial^2\Psi}{\partial x^2} = -k^2\Psi = -\frac{p^2}{\hbar^2}\Psi \implies p^2\Psi = -\hbar^2\frac{\partial^2\Psi}{\partial x^2} \tag{16}$$

$$\frac{\partial\Psi}{\partial t} = -i\omega\Psi = -i\frac{E}{\hbar}\Psi \implies E\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{17}$$

- using Eq. 16 and 17 in Eq. 15, one gets Schrödinger equation for a non-relativistic particle, with no spin, in a potential V:

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi} \tag{18}$$

The Schrödinger equation (1926)

Continuity equation and Quantum description:

- for some volume Vol. where no particles are created or destroyed the charge conservation is given by:

$$\underbrace{-\frac{\partial \rho}{\partial t}}_{\text{decrease of N of particles in Vol.}} = \underbrace{\boldsymbol{\nabla} \cdot \mathbf{j}}_{\text{N of particles leaving Vol.}} \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{19}$$

  where ρ is charge density and $\mathbf{j}$ is the current or the flux of ρ

- what is the connection with the above **continuity equation** in electromagnetism and a quantum mechanical description?
- need to find what are ρ and $\mathbf{j}$ in quantum mechanical formalism
- for this, start with a Schrödinger equation for a free particle ($V = 0$):

$$i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \Psi = 0 \tag{20}$$

- multiply Eq. 20 by Ψ^*, multiply the complex conjugate Eq. 20 by Ψ, and subtract the two

The Schrödinger equation (1926)

- subtraction of the two equations would be:

$$\Psi^*\left(i\hbar\frac{\partial\Psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\Psi\right)-\Psi\left(-i\hbar\frac{\partial\Psi^*}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\Psi^*\right)=0 \quad (21)$$

$$\Longrightarrow\ i\hbar\left(\Psi^*\frac{\partial\Psi}{\partial t}+\Psi\frac{\partial\Psi^*}{\partial t}\right)+\frac{\hbar^2}{2m}\left(\Psi^*\nabla^2\Psi-\Psi\nabla^2\Psi^*\right)=0 \quad (22)$$

$$\Longrightarrow\ \frac{\partial(\Psi\Psi^*)}{\partial t}+\boldsymbol{\nabla}\cdot\left(-\frac{\mathbf{i\hbar}}{\mathbf{2m}}\right)\left(\ ^*\boldsymbol{\nabla}\ -\ \boldsymbol{\nabla}\ ^*\right)=0 \quad (23)$$

$$\Longrightarrow\ \text{resembles}\ \frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{j}=0 \quad (24)$$

- from here:

$$\rho\equiv\Psi\Psi^*=|\Psi|^2,\mathbf{j}=-\frac{i\hbar}{2m}\left(\Psi^*\boldsymbol{\nabla}\Psi-\Psi\boldsymbol{\nabla}\Psi^*\right) \quad (25)$$

where ρ is the probability density and $\mathbf{j}$ is current

Plane wave example

- switching to natural units with $\hbar = c = 1$, for a plane wave:

$$\Psi = Ne^{i(\mathbf{p}\cdot\mathbf{r} - Et)} \implies \rho = |N|^2 \text{ and } \mathbf{j} = |N|^2 \frac{\mathbf{p}}{m} = |N|^2 \mathbf{v} \tag{26}$$

- the number or particles per unit volume is $|N|^2$
- for $|N|^2$ particles per unit volume moving at velocity $\mathbf{v}$, have $|N|^2\mathbf{v}$ passing through a unit area per unit time (particle flux)
- therefore $\mathbf{j}$ is a vector in the particle's direction with magnitude equal to the **flux**

The Klein-Gordon equation (1926)

- following the same spirit, Oscar Klein and Walter Gordon made an attempt to find QM equation which describes relativistic electron
- start from relativistic relation between energy and momentum for a free particle:

$$E^2 = p^2 + m^2 \tag{27}$$

- replace E and p with the corresponding operators:

$$\left(\widehat{E}\right)^2 \Psi = \left(\widehat{P}\right)^2 \Psi + m^2 \Psi \tag{28}$$

$$E \implies i\frac{\partial}{\partial t}, p_x \implies -i\frac{\partial}{\partial x}, p_y \implies -i\frac{\partial}{\partial y}, p_z \implies -i\frac{\partial}{\partial z} \tag{29}$$

$$\left(i\frac{\partial}{\partial t}\right)^2 \Psi = (-i\boldsymbol{\nabla})^2 \Psi + m^2 \Psi \tag{30}$$

- Klein-Gordon equation for a relativistic particle and no spin:

$$\frac{\partial^2 \Psi}{\partial t^2} = \nabla^2 \Psi - m^2 \Psi \tag{31}$$

The Klein-Gordon equation (1926)

- using $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$ can write down Klein-Gordon equation as:

$$\left(\partial^\mu \partial_\mu + m^2\right)\Psi = 0 \tag{32}$$

Problems with the Klein-Gordon equation:

- for plane wave solutions $\Psi = Ne^{i(\mathbf{p}\cdot\mathbf{r} - Et)}$ the Klein-Gordon equation gives:

$$-E^2\Psi = -|\mathbf{p}|^2\Psi - m^2\Psi \tag{33}$$

$$\Longrightarrow \; E = \pm\sqrt{|\mathbf{p}|^2 + m^2} \tag{34}$$

- this is the same result as one gets from Eq. 27
- historically, these negative solutions were viewed as problematic:
  - it implied no ground state in the atoms
  - transitions to lower energy states always possible

The Klein-Gordon equation (1926)

Problems with the Klein-Gordon equation:

- proceeding as before to calculate the probability and current densities:

$$\rho = i\left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right) \text{ and } \mathbf{j} = i(\Psi^* \boldsymbol{\nabla} \Psi - \Psi \boldsymbol{\nabla} \Psi^*) \tag{35}$$

- for a plane wave: $\Psi = Ne^{i(\mathbf{p}\cdot\mathbf{r} - Et)}$

$$\rho = 2E|N|^2 \text{ and } \mathbf{j} = |N|^2 \mathbf{p} \tag{36}$$

- $\Longrightarrow$ particle densities are proportional to E: can also be negative
- how can the probability be negative? No interpretation could be made at that time

The Dirac equation (1928)

- to summarize, at the time there were two main problems with the KG equation:
  - negative energy solutions
  - negative particle densities associated with these solutions
- in fact, now in Quantum Field Theory these problems are overcome and the KG equation is used to describe spin-0 particles, e.g. pions
- back in the day, these problems led to new developments:

- they motivated Dirac to search for a different formulation of relativistic quantum mechanics in which all particle densities are positive
- the resulting wave equation had solutions which not only solved this problem but also fully described the intrinsic spin and magnetic moment of the electron!

The Dirac equation (1928)

- Schrödinger equation:

$$-\frac{1}{2m}\nabla^2\Psi = i\frac{\partial\Psi}{\partial t} \tag{37}$$

  - first order in $\frac{\partial}{\partial t}$
  - second order in $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

- Klein-Gordon equation:

$$\left(\partial^\mu\partial_\mu + m^2\right)\Psi = 0 \tag{38}$$

  - second order throughout

- Dirac looked for an alternative which was first order throughout:

$$\widehat{H}\Psi = (\alpha\cdot\mathbf{p} + \beta m)\Psi = i\frac{\partial\Psi}{\partial t} \tag{39}$$

where $\widehat{H}$ is the Hamiltonian operator and $\mathbf{p} = -i\boldsymbol{\nabla}$

The Dirac equation (1928)

- writing down Eq. 39 in full and squaring it leads to:

$$\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\Psi=\left(i\frac{\partial}{\partial t}\right)\Psi \tag{40}$$

$$\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\Psi=-\frac{\partial^2\Psi}{\partial t^2} \tag{41}$$

$$-\frac{\partial^2\Psi}{\partial t^2}=-\alpha_x^2\frac{\partial^2\Psi}{\partial x^2}-\alpha_y^2\frac{\partial^2\Psi}{\partial y^2}-\alpha_z^2\frac{\partial^2\Psi}{\partial z^2}+\beta^2m^2\Psi \tag{42}$$

$$-(\alpha_x\alpha_y+\alpha_y\alpha_x)\frac{\partial^2\Psi}{\partial x\partial y}-(\alpha_y\alpha_z+\alpha_z\alpha_y)\frac{\partial^2\Psi}{\partial y\partial z}-(\alpha_z\alpha_x+\alpha_x\alpha_z)\frac{\partial^2\Psi}{\partial z\partial x} \tag{43}$$

$$-(\alpha_x\beta+\beta\alpha_x)m\frac{\partial\Psi}{\partial x}-(\alpha_y\beta+\beta\alpha_y)m\frac{\partial\Psi}{\partial y}-(\alpha_z\beta+\beta\alpha_z)m\frac{\partial\Psi}{\partial z} \tag{44}$$

The Dirac equation (1928)

- for this to be a reasonable formulation of relativistic QM, a free particle must also obey $E^2 = \mathbf{p}^2 + m^2$, i.e. it must satisfy the Klein-Gordon equation:

$$-\frac{\partial^2 \Psi}{\partial t^2} = -\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial z^2} + m^2 \Psi \tag{45}$$

- hence for the Dirac equation to be consistent with the KG equation require:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1 \tag{46}$$

$$\alpha_j \beta + \beta \alpha_j = 0 \tag{47}$$

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 (j \neq k) \tag{48}$$

- obviously, α_i and β cannot be numbers: require 4 mutually anti-commuting matrices
- must be (at least) 4×4 matrices

The Dirac equation (1928)

- consequently, the wave-function must be a four-component Dirac spinor: it has new degrees of freedom as a result of introducing an equation that is first order in time/space derivatives

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \tag{49}$$

- for the Hamiltonian $\widehat{H} = (\alpha \cdot \mathbf{p} + \beta m)\Psi = i\frac{\partial \Psi}{\partial t}$ to be Hermitian:

$$\alpha_x = \alpha_x^\dagger; \alpha_y = \alpha_y^\dagger; \alpha_z = \alpha_z^\dagger; \beta = \beta^\dagger \tag{50}$$

  i.e. requires four anti-commuting Hermitian 4×4 matrices
- at this point it is convenient to introduce an explicit representation for α, β
- it should be noted that physical results do not depend on the particular representation: everything is in the commutation relations

Pauli spin matrices

A convenient choice is based on the Pauli spin matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \tag{51}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{52}$$

The matrices are Hermitian and anti-commute with each other.

Dirac equation: Probability density and current

- let's get back to the probability density and current which were in trouble with the KG equation
- Dirac equation:

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\Psi = \left(i\frac{\partial}{\partial t}\right)\Psi \tag{53}$$

- its Hermitian conjugate:

$$+i\frac{\partial\Psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial\Psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial\Psi^\dagger}{\partial z}\alpha_z^\dagger + m\Psi^\dagger\beta^\dagger = -\left(i\frac{\partial}{\partial t}\right)\Psi^\dagger \tag{54}$$

- compute $\Psi^\dagger\times$Eq. 53$-$Eq. 54$\times\Psi$ taking into account that α, β are Hermitian, and taking into account that

$$\Psi^\dagger\alpha_x\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi^\dagger}{\partial x}\alpha_x\Psi \equiv \frac{\partial\left(\Psi^\dagger\alpha_x\Psi\right)}{\partial x} \tag{55}$$

Dirac equation: Probability density and current

We get the continuity equation:

$$\boldsymbol{\nabla} \cdot \left(\Psi^\dagger \boldsymbol{\alpha} \Psi\right) + \frac{\partial\left(\Psi^\dagger \Psi\right)}{\partial t} = 0 \tag{56}$$

where $\Psi^\dagger = (\Psi_1^*, \Psi_2^*, \Psi_3^*, \Psi_4^*)$

- the probability density and current are:

$$\rho = \Psi^\dagger \Psi, \quad \mathbf{j} = \Psi^\dagger \boldsymbol{\alpha} \Psi \tag{57}$$

  where $\rho = \Psi^\dagger \Psi = |\Psi_1|^2 + |\Psi_2|^2 + |\Psi_3|^2 + |\Psi_4|^2 > 0$

- unlike the KG equation, the Dirac equation has probability densities which are **always positive**
- the solutions to the Dirac equation are **the four component Dirac Spinors**. A great success of the Dirac equation is that these components naturally give rise to the property of intrinsic spin

Covariant notation: the Dirac γ matrices

- the Dirac equation can be written more elegantly by introducing the four Dirac gamma matrices:

$$\gamma^0 \equiv \beta; \gamma^1 \equiv \beta\alpha_x; \gamma^2 \equiv \beta\alpha_y; \gamma^3 \equiv \beta\alpha_z \quad (58)$$

- multiplying Eq. 53 by $-\beta$ one gets:

$$\left(i\beta\alpha_x\frac{\partial}{\partial x} + i\beta\alpha_y\frac{\partial}{\partial y} + i\beta\alpha_z\frac{\partial}{\partial z} - \beta^2 m\right)\Psi = -\left(i\beta\frac{\partial}{\partial t}\right)\Psi \quad (59)$$

$$\Longrightarrow \; i\gamma^1\frac{\partial\Psi}{\partial x} + i\gamma^2\frac{\partial\Psi}{\partial y} + i\gamma^3\frac{\partial\Psi}{\partial z} - m\Psi = -i\gamma^0\frac{\partial\Psi}{\partial t} \quad (60)$$

- using $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ can rewrite as:

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0 \quad (61)$$

- **the Dirac gamma matrices are not four-vectors: they are constant matrices which remain invariant under a Lorentz transformation**

The Dirac equation: solutions

- consider a particle at rest, $p = 0$:

$$\left(i\gamma^0 \frac{\partial}{\partial t} - m\right)\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}, \text{ where } \gamma^0 = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix} \tag{62}$$

- spinor Ψ naturally splits into two 2-component bi-spinors: $\Psi \equiv \begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix}$, and:

$$i\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix} = m\begin{pmatrix}\Psi_A\\ \Psi_B\end{pmatrix} \tag{63}$$

$$\Longrightarrow \quad i\frac{\partial \Psi_A}{\partial t} = m\Psi_A, \quad i\frac{\partial \Psi_B}{\partial t} = -m\Psi_B \tag{64}$$

- the solutions are written as a function of the bi-spinors u_A and u_B:

$$\Psi_A(t) = u_A e^{-imt}, \quad E > 0 : \text{positive energy solutions} \tag{65}$$

$$\Psi_B(t) = u_B e^{imt}, \quad E < 0 : \text{negative energy solutions} \tag{66}$$

The Dirac equation: solutions

- going back to Eq. 63:

$$\begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = m \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{67}$$

- since left-hand side is diagonal, we can find decoupled solutions for u_A and u_B, and choose as a set of eigenvectors:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } E = +m \tag{68}$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } E = -m \tag{69}$$

The Dirac equation: solutions

- putting all together, for a particle at rest we find:

$$\Psi_0^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}; \Psi_0^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \text{ with positive energy} \tag{70}$$

$$\Psi_0^{(3)} = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}; \Psi_0^{(4)} = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt} \text{ with negative energy} \tag{71}$$

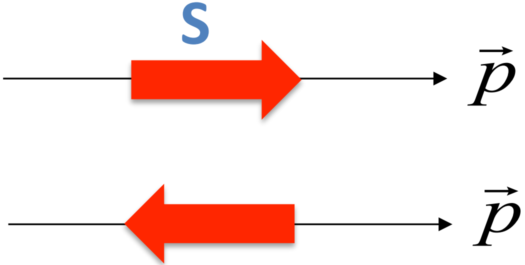
- four solutions: two with positive energy and two with negative;

The Dirac equation: solutions

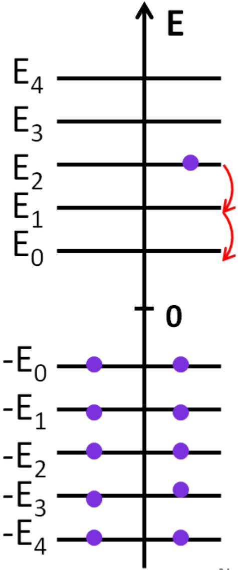
- the fact that there are two identical fermions with the same energy implies that there is another quantum number that allows to distinguish them, the helicity. The corresponding operator is the operator projecting the spin on the direction of motion:

h=+1 positive helicity

h=−1 negative helicity



Dirac's explanation for negative energy solutions



The atoms are observed to be stable. When an electron is on a high energy level, it undergoes transitions down to the state of lowest positive energy not yet occupied by 2 electrons.

To save his equation, Dirac makes the hypothesis that all the states of negative energy are already occupied by 2 electrons, preventing another electron to reach these states.

All these electrons filling the negative states form what was called the Dirac sea.

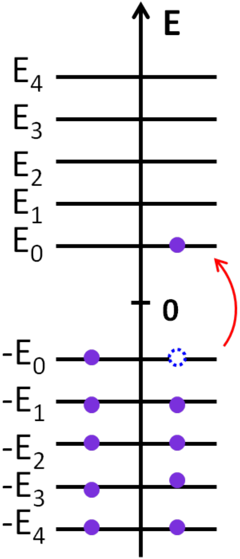
Dirac's explanation for negative energy solutions

But what happens if a sufficient energy is provided to an electron of the sea? It appears like a hole in the sea:

Missing $q = -e \implies$ Presence of $q = +e$
Missing $E < 0 \implies$ Presence of $E > 0$

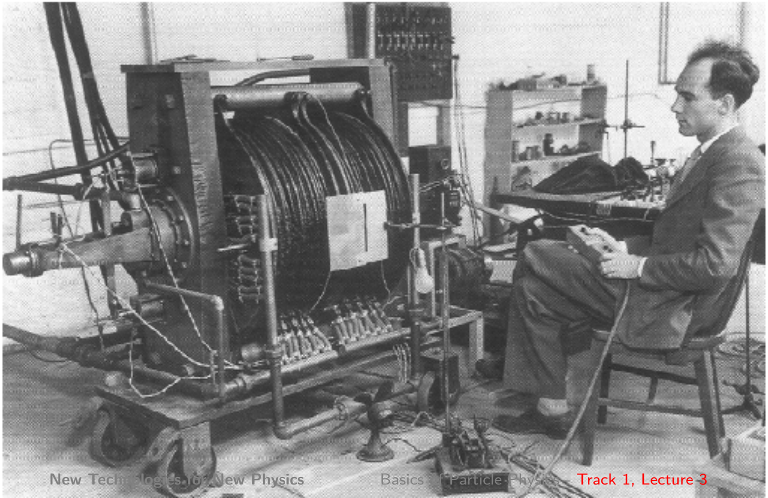
A hole in the electron sea at energy level $E < 0$ looks like an ordinary particle with charge $q = +e$ and energy $-E > 0$!

Would such positive electron exist? Then they should be identical to the electrons, except their charge.



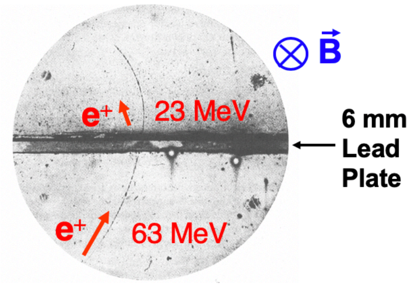
Discovery of anti-electron (1932)

The mystery of negative energy solutions of Dirac's equation persisted until 1932 when C. Anderson discovered a new particle seeming identical to electron but with opposite charge.



Discovery of anti-electron (1932)

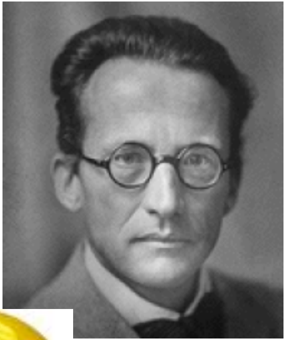
- he used a cloud chamber - a tube full of super-saturated liquid. Charged particles passing through ionize it, which then seeds an ion trail that can be photographed
- applied a uniform magnetic field
- he observed the tracks of a positively charged particle for which the energy losses in the Pb-plate were not compatible with those of a proton
- on the contrary, the track looked exactly like an electron, except the charge



This was the observation of the first antiparticle, the anti-electron, called positron

Discovery of anti-electron (1932)





- 1933: Dirac together with Schrödinger receives the Nobel prize
- 1936: Anderson, at the age of 31, becomes the second youngest Nobel prize winner

The antiparticles

Feynman-Stückelberg Interpretation for $E < 0$ (1940):

- the story of the sea of electrons was not very satisfactory (infinite negative charge of the Universe!)
- new hypothesis supported by the positron observation
- **To each particle of mass m and charge q corresponds an antiparticle of mass m and charge $-q$**

- Indeed, the $E < 0$ solution can be seen instead as $-Et \implies E(-t)$
- corresponds to a particle of positive energy E with time inversed
- nowadays, there is an antiparticle associated to each known particle, making the positron discovery one of the milestones of contemporary particle physics