Particles and their interactions

Lesya Shchutska

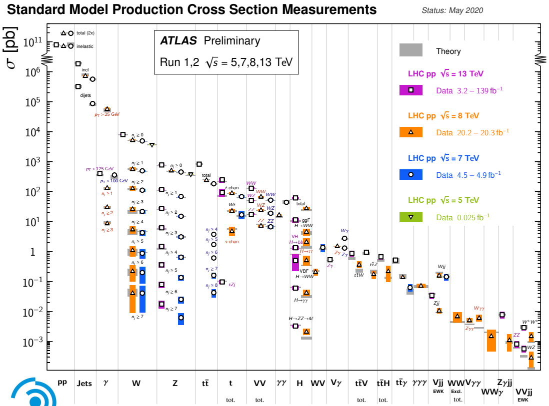
EPFL

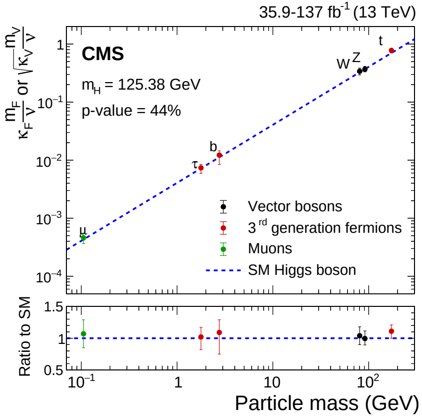
Questions on the lecture

- after going through the material of the lecture, post at least one question through the anonymous google form:
  - https://forms.gle/pGMV5PbCZKCwtrM29
- we will go over the questions during the live lecture

Cross sections and decay rates

- all calculations in particle physics revolve around particle **interactions** and **decays**, i.e. **transitions between states**
  - these are experimental observables of particle physics





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Cross sections and decay rates

- we can calculate transition rates from **Fermi's Golden Rule**:

$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)} \quad (1)$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$: **not Lorentz invariant**

T_{fi} is transition **matrix element (ME)**:

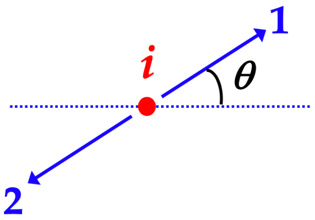
$$T_{fi} = \langle f|\widehat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\widehat{H}|j\rangle \; \langle j|\widehat{H}|i\rangle}{E_i - E_j} + \ldots \quad (2)$$

$\widehat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

- rates depend on **matrix element** (=fundamental particle physics) and **density of states** (=kinematics)

Particle decay rates



- consider the two-body decay $i \to 1 + 2$
- want to calculate the decay rate in first order perturbation theory using plane-wave description of the particles (Born approximation)

$$\psi_1 = N e^{i(\vec{p}\cdot\vec{r} - ET)} = N e^{-ip\cdot x} \tag{3}$$

where N is the normalization and $p \cdot x = p^\mu x_\mu$

Particle decay rates

For decay rate calculation need to know (in a Lorentz invariant form):

1. wave-function normalisation
2. transition matrix element from perturbation theory
3. expression for the density of states

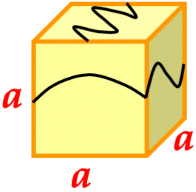
First consider wave-function normalization:

- non-relativistic formulation: normalized to one particle in a cube of side a:

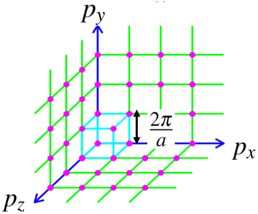
$$\int \Psi\Psi^* \mathrm{d}V = N^2 a^3 = 1 \implies N^2 = 1/a^3 \tag{4}$$

Non-relativistic phase space

- apply boundary conditions: $\vec{p} = \hbar \vec{k}$
- wave-function vanishing at box boundaries $\Longrightarrow$ quantized particle momenta:



$$p_x = \frac{2\pi n_x}{a}; \quad p_y = \frac{2\pi n_y}{a}; \quad p_z = \frac{2\pi n_z}{a}; \tag{5}$$



- volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V} \tag{6}$$

- normalizing to one particle/unit volume gives **number of states** in element $\mathrm{d}^3\vec{p} = \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$:

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3/V} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \tag{7}$$

Non-relativistic phase space

- therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}E}\right|_{E_f} \tag{8}$$

- transformations of the elements in Eq. 8 using Eq. 7:

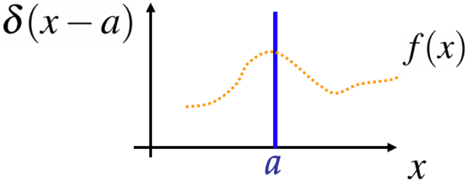
$$\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|} = \frac{1}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}}{\mathrm{d}|\vec{p}|} = \frac{4\pi p^2\mathrm{d}p}{(2\pi)^3\mathrm{d}p} = \frac{4\pi p^2}{(2\pi)^3} \tag{9}$$

$$E^2 = p^2 + m^2 \implies 2E\mathrm{d}E = 2p\mathrm{d}p \implies \frac{\mathrm{d}p}{\mathrm{d}E} = \frac{E}{p} = \frac{1}{\beta} \tag{10}$$

$$\implies \rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \frac{1}{\beta} \tag{11}$$

Dirac δ-function

- in the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: "infinitely narrow spike of unit area"



$$\int_{-\infty}^{+\infty} \delta(x-a)\mathrm{d}x = 1 \qquad (12)$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\mathrm{d}x = f(a) \qquad (13)$$

- any function with the above properties can represent $\delta(x)$, e.g.

$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)} \qquad (14)$$

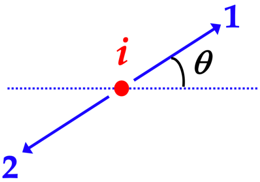
- an infinitesimally narrow Gaussian

Dirac δ-function: use example

- in relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $i \to 1 + 2$:

$$\int \ldots \delta(E_i - E_1 - E_2)\mathrm{d}E \text{ and } \int \ldots \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\mathrm{d}^3\vec{p} \qquad (15)$$

express energy and momentum conservation



Dirac δ-function of a function

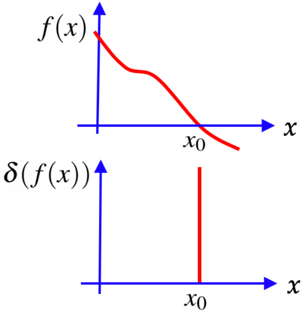
An expression for the δ-function of a function $\delta(f(x))$:

- start from the definition of a delta function:

$$\int_{y_1}^{y_2} \delta(y)\mathrm{d}y = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

- now express in terms of $y = f(x)$, where $f(x_0) = 0$, and then change variables:

$$\int_{x_1}^{x_2} \delta(f(x))\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$



Dirac δ-function of a function

- from properties of the δ-function (i.e. here only non-zero at x_0):

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| \int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- rearranging and expressing RHS as a δ-function:

$$\int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)\mathrm{d}x \tag{16}$$

$$\Longrightarrow \boxed{\delta(f(x)) = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}^{-1} \delta(x - x_0)} \tag{17}$$

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f) \tag{18}$$

- rewrite the expression for density of states using a δ-function:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E - E_i)\mathrm{d}E \text{ since } E_f = E_i \tag{19}$$

  Note: integrating over all final state energies but energy conservation now taken into account explicitly by δ-function

- hence the golden rule becomes an integral over all "allowed" final states of **any energy**:

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\mathrm{d}n \tag{20}$$

The Golden Rule revisited

- for $\mathrm{d}n$ in a two-body decay, only need to consider one particle, as momentum conservation fixes the other:

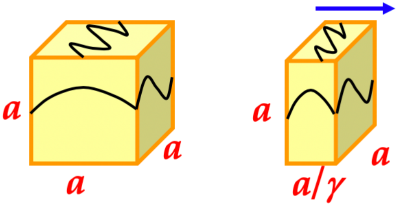
$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3} \tag{21}$$

- however, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-function:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{E conservation}} \underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\vec{p}\text{ conservation}} \underbrace{\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3}}_{\text{density of states}} \tag{22}$$

Lorentz invariant phase space

- in non-relativistic QM normalize to one particle/unit volume: $\int \Psi^* \Psi \mathrm{d}V = 1$
- when considering relativistic effects, volume contracts by $\gamma = E/m$



- particle density therefore increases by $\gamma = E/m$: hence a relativistic invariant wave-function normalisation needs to be proportional to E particles per unit volume

Lorentz invariant phase space

- usual convention: normalize to $2E$ particles/unit volume: $\int \Psi'^{*}\Psi' \mathrm{d}V = 2E$
- hence $\Psi' = (2E)^{1/2}\Psi$ is normalized to $2E$ per unit volume
- define Lorentz invariant matrix element, M_{fi}, in terms of the wave-functions normalized to $2E$ particles per unit volume:

$$\boxed{\mathrm{M}_{fi} = \left\langle \Psi'_1 \cdot \Psi'_2 \ldots \middle| \widehat{H} \middle| \ldots \Psi'_{n-1} \cdot \Psi'_n \right\rangle = (2E_1 \cdot 2E_2 \ldots 2E_{n-1} \cdot 2E_n)^{1/2} T_{fi}} \qquad (23)$$

Two-body decay

$$M_{fi} = \left\langle \Psi_1' \cdot \Psi_2' \middle| \widehat{H} \middle| \Psi_i' \right\rangle = (2E_i \cdot 2E_1 \cdot 2E_2)^{1/2} \left\langle \Psi_1 \cdot \Psi_2 \middle| \widehat{H} \middle| \Psi_i \right\rangle \quad (24)$$

$$= (2E_i \cdot 2E_1 \cdot 2E_2)^{1/2} T_{fi} \quad (25)$$

- expressing T_{fi} in terms of M_{fi} gives:

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2} \quad (26)$$

- M_{fi} uses relativistically normalized wave-functions and it is Lorentz invariant
- $\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz invariant phase space for each final state particle: the factor of $2E$ arises from the wave-function normalization

Two-body decay

- this form of Γ_{fi} is simply a rearrangement of the original equation but the integral is now frame independent
- Γ_{fi} is inversely proportional to E_i, the energy of the decaying particle: this is what's expected from the time dilation
- energy and momentum conservation is in the δ-functions

Decay rate calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \quad (27)$$

- since the integral is Lorentz invariant it can be evaluated in any frame: the C.o.M. frame is most convenient
- in the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0 \implies$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2} \quad (28)$$

- integrating over $\vec{p}_2$ using the δ-function:

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3\vec{p}_1}{4E_1 E_2} \quad (29)$$

Decay rate calculations

- now $E_2^2 = m_2^2 + |\vec{p}_1|^2$ since the δ-function imposes $\vec{p}_2 = -\vec{p}_1$
- writing $\mathrm{d}^3\vec{p}_1 = p_1^2 \mathrm{d}p_1 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega$

$$\Longrightarrow \Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega}{E_1 E_2} \qquad (30)$$

Decay rate calculations

- which can be written in the form:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \tag{31}$$

where $g(p_1) = p_1^2/(E_1E_2) = p_1^2\left(m_1^2 + p_1^2\right)^{-1/2}\left(m_2^2 + p_1^2\right)^{-1/2}$ (32)

and $f(p_1) = m_i - \left(m_1^2 + p_1^2\right)^{1/2} - \left(m_2^2 + p_1^2\right)^{1/2}$ (33)

- note that $\delta(f(p_1))$ imposes energy conservation
- $f(p_1) = 0$ determines the C.o.M. momenta of the two decay products, i.e. $f(p_1) = 0$ for $p_1 = -p_2 = p^*$

Decay rate calculations

- integrating Eq. 31 using the property of δ-function from Eq. 17:

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}} \int g(p_1)\delta(p - p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}} \qquad (34)$$

where p^* is the value for which $f(p^*) = 0$

Decay rate calculations

- now we need to evaluate $\mathrm{d}f/\mathrm{d}p_1$:

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{\left(m_1^2+p_1^2\right)^{1/2}} - \frac{p_1}{\left(m_2^2+p_1^2\right)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2} \tag{35}$$

$$\Longrightarrow \Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} \mathrm{d}\Omega \tag{36}$$

$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{p_1}{E_1+E_2}\right|_{p_1=p^*} \mathrm{d}\Omega \tag{37}$$

- but from $f(p_1)=0$, i.e. energy conservation: $E_1+E_2=m_i$:

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 \mathrm{d}\Omega \tag{38}$$

Decay rate calculations

- in the particle's rest frame $E_i = m_i$:

$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p^*}|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \mathrm{d}\Omega \tag{39}$$

valid for all two-body decays

- p^* can be obtained from $f(p_1) = 0$:

$$\left(m_1^2 + p^{*2}\right)^{1/2} + \left(m_2^2 + p^{*2}\right)^{1/2} = m_i \tag{40}$$

$$\implies p^* = \frac{1}{2m_i}\sqrt{\left[m_i^2 - (m_1 + m_2)^2\right]\left[m_i^2 - (m_1 - m_2)^2\right]} \tag{41}$$

Cross section definition

$$\sigma = \frac{\text{number of interactions per unit time per target}}{\text{incident flux}} \qquad (42)$$

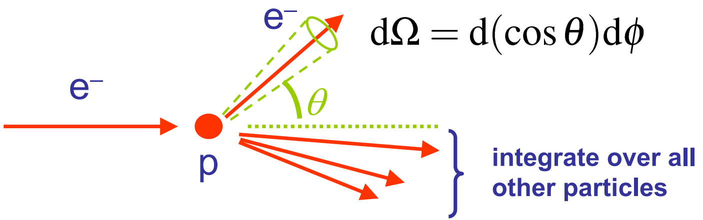
where flux = number of incident particles/unit area/unit time

- the "cross section", σ, can be thought of as the effective cross-sectional area of the target particles for the interaction to occur
- in general, this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

Differential cross section definition

Differential cross section: $\dfrac{d\sigma}{d\ldots}$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\text{number of interactions per unit time per target into } d\Omega}{\text{incident flux}}} \tag{43}$$



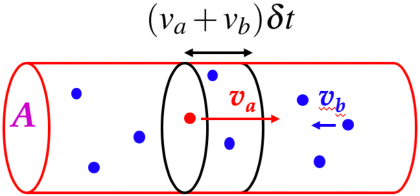
$$d\Omega = d(\cos\theta)d\phi$$

with $\boxed{\sigma = \int \frac{d\sigma}{d\Omega} d\Omega}$

Cross section: example

- consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume:

in time δt a particle of type a traverses region containing $n_b(v_a + v_b)A\delta t$ particles of type b



- interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a + v_b)A\delta t$ particles of type b:

$$\frac{n_b(v_a + v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma, \quad v = v_a + v_b \tag{44}$$

$\Longrightarrow$ rate per particle of type a is $n_b v \sigma$

Cross section: example

- consider volume V, total reaction rate:

$$(n_b v \sigma) \cdot (n_a V) = (n_b V)(n_a v)\sigma = N_b \phi_a \sigma \tag{45}$$

- i.e. **rate = flux** $\times$ **number of targets** $\times$ **cross section**

Cross section calculations

- consider scattering process $1 + 2 \to 3 + 4$
- start from Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3} \tag{46}$$

where T_{fi} is the transition matrix for a normalization of 1/unit volume

- Rate/Volume = (flux of 1)×(number density of 2)×$\sigma = n_1(v_1 + v_2) \times n_2 \times \sigma$
- for 1 target particle per unit volume, rate $= (v_1 + v_2)\sigma \implies \sigma = \dfrac{\Gamma_{fi}}{v_1 + v_2}$

$$\sigma = \underbrace{\frac{(2\pi)^4}{v_1 + v_2}} \int \underbrace{|T_{fi}|^2} \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \underbrace{\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}} \tag{47}$$

these parts are not Lorentz invariant

Cross section calculations

- to obtain a Lorentz invariant form use wave-functions normalised to $2E$ particles per unit volume: $\Psi' = (2E)^{1/2}\Psi$
- again define L.I. matrix element $M_{fi} = (2E_1 2E_2 2E_3 2E_4)^{1/2} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{\mathrm{d}^3\vec{p}_3}{2E_3} \frac{\mathrm{d}^3\vec{p}_4}{2E_4} \quad (48)$$

- the integral is now written in a Lorentz invariant form
- the quantity $F = 2E_1 2E_2 (v_1 + v_2)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz invariant:

$$F = 4\left[(p_1^{\mu} p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2} \quad (49)$$

- consequently cross section is a Lorentz invariant quantity

Two special cases of Lorentz Invariant Flux

1. Center-of-Mass frame:

$$F = 4E_1E_2(v_1 + v_2) \tag{50}$$

$$= 4E_1E_2\left(\frac{|\vec{p}^*|}{E_1} + \frac{|\vec{p}^*|}{E_2}\right) \tag{51}$$

$$= 4|\vec{p}^*|(E_1 + E_2) \tag{52}$$

$$= 4|\vec{p}^*|\sqrt{s} \tag{53}$$

2. Target particle (particle 2) at rest:

$$F = 4E_1E_2(v_1 + v_2) \tag{54}$$

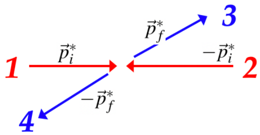
$$= 4E_1m_2v_1 \tag{55}$$

$$= 4E_1m_2\frac{|\vec{p}_1|}{E_1} \tag{56}$$

$$= 4m_2|\vec{p}_1| \tag{57}$$

$2 \to 2$ body scattering in C.o.M. frame

We will now apply above Lorentz invariant formula for the interaction cross section to the most common cases here. First consider $2 \to 2$ scattering in C.o.M. frame



- start from:

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4} \quad (58)$$

- here $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$

$$\Longrightarrow \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta\left(\sqrt{s}-E_3-E_4\right)\delta^3(\vec{p}_3+\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4} \quad (59)$$

$2 \to 2$ body scattering in C.o.M. frame

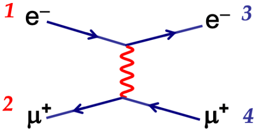
- the integral is exactly the same integral that appeared in the particle decay calculation with m_i replaced by $\sqrt{s}$:

$$\implies \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 \mathrm{d}\Omega^* \quad (60)$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 \mathrm{d}\Omega^*} \quad (61)$$

$2 \to 2$ body scattering in C.o.M. frame

- in the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$:



$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 \mathrm{d}\Omega^* \qquad (62)$$

- for calculating the total cross section (which is Lorentz invariant) the result on the previous page is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \mathrm{d}\Omega^* \qquad (63)$$

  because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M. frame

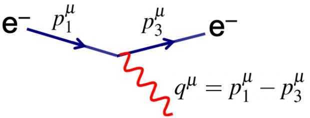
- for the last calculation in this section, we need to find a L.I. expression for $\mathrm{d}\sigma$

$2 \to 2$ body scattering in C.o.M. frame

- start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam t, i.e. square of the 4-momentum transfer,

$$t = q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$$

In C.o.M. frame:

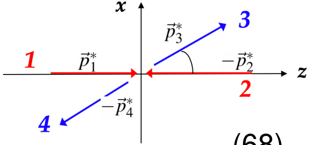


$$p_1^{*\mu} = (E_1^*, 0, 0, |\vec{p}_1^*|) \quad (64)$$

$$p_3^{*\mu} = (E_3^*, |\vec{p}_3^*| \sin\theta^*, 0, |\vec{p}_3^*| \cos\theta^*) \quad (65)$$

$$p_1^\mu p_{3\mu} = E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*| \cos\theta^* \quad (66)$$

$$t = m_1^2 + m_3^2 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*| \cos\theta^* \quad (67)$$



$$\Longrightarrow \mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*) \quad (68)$$

$$\Longrightarrow \mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \frac{\mathrm{d}t\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|} \quad (69)$$

$$\Longrightarrow \mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \frac{1}{2 \cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t \quad (70)$$

$2 \to 2$ body scattering in C.o.M. frame

- finally, integrating over $\mathrm{d}\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$):

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s |\vec{p}_i^*|^2} |M_{fi}|^2} \qquad (71)$$

- all quantities in this expression are L.I. and therefore, it applies to any rest frame. Also note that $|\vec{p}_1^*|^2$ is a constant, fixed by energy/momentum conservation:

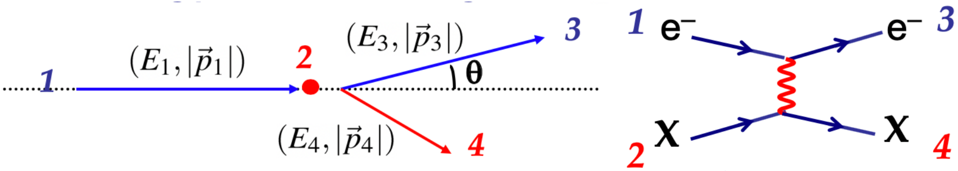
$$|\vec{p}_1^*|^2 = \frac{1}{4s}\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right] \qquad (72)$$

- as an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$, e.g. electron or neutrino scattering
- in this limit $|\vec{p}_1^*|^2 = (s - m_2)^2/(4s)$ and

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi (s - m_2^2)^2} |M_{fi}|^2} \qquad (73)$$

$2 \to 2$ body scattering in lab. frame

- the other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- first, take the case of elastic scattering at high energy where the mass of the incoming particles is neglected: $m_1 = m_3 = 0, m_2 = m_4 = M$



- wish to express the cross section in terms of scattering angle of the e^-: $\mathrm{d}\Omega = 2\pi \mathrm{d}(\cos\theta)$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \tag{74}$$

$2 \to 2$ body scattering in lab. frame

- four-momenta of particles:

$$p_1 = (E_1, 0, 0, E_1) \tag{75}$$
$$p_2 = (M, 0, 0, 0) \tag{76}$$
$$p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta) \tag{77}$$
$$p_4 = (E_4, \vec{p}_4) \tag{78}$$

$$\implies t = (p_1 - p_3)^2 = -2 p_1 \cdot p_3 = -2 E_1 E_3 (1 - \cos\theta) \tag{79}$$

- from $(E, \vec{p})$ conservation $p_1 + p_2 = p_3 + p_4 \implies$ can express t in terms of particles 2 and 4:

$$t = (p_2 - p_4)^2 = 2M^2 - 2 p_2 \cdot p_4 = 2M^2 - 2ME_4 \tag{80}$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3) \tag{81}$$

- E_1 is a constant (the energy of the incoming particle), hence:

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \tag{82}$$

$2 \to 2$ body scattering in lab. frame

- equating the two expressions for t gives:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta} \quad (83)$$

$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M \left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M} \quad (84)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s-M^2)^2}|M_{fi}|^2 \quad (85)$$

- using $s = (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 = M^2 + 2ME_1$, as $p_1^2 = 0$, gives $(s - M^2) = 2ME_1$

$$\Longrightarrow \boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2} \text{ in the limit of } m_1 \to 0 \quad (86)$$

$2 \to 2$ body scattering in lab. frame

- in this equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta} \quad (87)$$

- giving

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1 \cos\theta}\right)^2 |M_{fi}|^2 \quad (88)$$

General form for $2 \to 2$ body scattering in lab. frame:

- in case the mass m_1 cannot be neglected, after a similar procedure:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{1}{p_1 m_1}\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta}|M_{fi}|^2 \quad (89)$$

There is only one independent variable, θ, from conservation of energy:

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2} \quad (90)$$

Summary

Used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element (wave-functions normalized to 2E/Volume)

Main results:

- particle decay:

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \mathrm{d}\Omega \tag{91}$$

where p^* is a function of particle masses:

$p^* = \frac{1}{2m_i}\sqrt{\left[m_i^2 - (m_1 + m_2)^2\right]\left[m_i^2 - (m_1 - m_2)^2\right]}$

- scattering cross section in C.o.M. frame:

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 \mathrm{d}\Omega^* \tag{92}$$

Summary

- invariant differential cross section (valid in all frames):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2} \int |M_{fi}|^2 \tag{93}$$

where $|\vec{p}_i^*|^2 = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right]$

- differential cross section in the lab. frame $(m_1 = 0)$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \Leftrightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2 \tag{94}$$

- differential cross section in the lab. frame $(m_1 \neq 0)$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{|\vec{p}_1|m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2 \tag{95}$$

with $E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}$

Summary

- Have now dealt with kinematics of particle decays and cross sections
- The fundamental particle physics is in the matrix element
- The above equations are the basis for all calculations that follow